NOTES

Periodic Points of the Open-Tent Function

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Given a function $f : S \to S$, it is of great interest in the field of dynamical systems to figure out which points in the set S are eventually sent back to themselves through repeated applications of f. More precisely, people like to know which points x and which positive integers n have the property that $f^n(x) = x$, where f^n denotes the nth iteration of f. Such a point x is called a *periodic point*, and the smallest such n is called the *prime period* of x. (Note that this does not require that n be a prime number.)

According to a famous theorem by Li and Yorke [**8**], for a continuous function f on a line or a closed interval S, if f has a point of prime period 3, then f has a point of prime period n for every n. This amazing result turns out to be a special case of the even more amazing Sarkovskii theorem [**4**, Ch. 11].

To construct a simple example of a continuous function with a point of prime period 3 on the unit interval, we choose $0 \to 1/2 \to 1 \to 0$ as our 3-cycle and connect the points (0, 1/2), (1/2, 1), and (1, 0) by a piecewise-linear function

$$f(x) = \begin{cases} x + 1/2 & \text{if } 0 \le x < 1/2, \\ 2 - 2x & \text{if } 1/2 \le x \le 1. \end{cases}$$

We call f the *open-tent function*. Its graph is given in FIGURE 1.

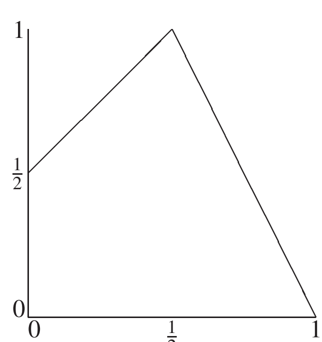

Figure 1 The open-tent function

The open-tent function f is well known and is used as a simple example to illustrate that prime period 3 implies chaos [**1**, p. 248; **3**, p. 135; **6**]. However, knowing the existence of points with various periods and actually finding them are two different matters. In a nice note, David Sprows [**10**] uses binary expansions to construct for the open-tent function f a point of prime period n for each positive integer n. However, with this method, information about the orbits of f is far from clear. The method we present in this Note gives a far more explicit picture.

The *orbit* of x is the set $\{x, f(x), f^2(x), f^3(x), \ldots\}$. For instance, the orbit of $2/3$ consists of a single point, while the orbit of $1/3$ includes a single additional point, $5/6$. Clearly, the orbit is finite if x is periodic.

Our key for giving a precise description of the orbits is through a method called *encoding* and *decoding*. One might view a binary expansion for any $x \in [0, 1]$ as an identification number, or "ID," for x. Our strategy is to provide each $x \in [0, 1]$ with a different infinite sequence of zeros and ones as its new ID; this is called *encoding*. With this new ID (encoding), we can know the orbital information of the open-tent function f much better. For example, we can tell *how many* points of period n there are and how we can locate *all* of them. We can also locate many other points with interesting orbital features, such as a point that stays obediently in the interval $[1/2, 1]$ for every single iteration of f, escaping exactly once on the one-millionth time.

The open-tent function is an example of a *dynamical system* (S, f), a set S together with a function f from S back to itself. The open-tent example, where the set is $[0, 1]$ would be written $([0, 1], f)$. The key idea of this note is as follows: First, we use a "digital (or symbolic) model" to encode the open-tent function system $([0, 1], f)$, namely, the well-known symbolic dynamical system (G, σ), called the *golden-mean shift*. Then we investigate the encoded orbital information of $([0, 1], f)$ in (G, σ) which is much easier to handle digitally. Finally, we decode the information obtained from (G, σ) back to the system $([0, 1], f)$ in the same way that a CD player decodes its digital codes back into music.

This approach connects many interesting topics in undergraduate mathematics, such as the golden mean, Fibonacci and Lucas numbers, directed graphs, matrices, binary expansions, and coding. Our technique is standard in the field of dynamical systems [**1, 4, 5, 7, 9**], but we provide a rigorous and complete coding algorithm for the open-tent function, including the coding for the numbers of the form $j/2^n$ (the boundary points that arise upon repeatedly bisecting the unit interval), which has previously been unavailable to students.

Unlike the *tent function* (obtained by replacing $x + 1/2$ by $2x$ for $0 \le x < 1/2$ in the definition of f), which is mentioned in almost every dynamical system text and utilizes all 0-1 sequences for its coding, the open-tent function gives us an elementary yet nontrivial example of coding in terms of a proper subset of the set of all 0-1 sequences, as well as a simple yet rich application of symbolic dynamics—a fast-growing branch of modern mathematics [**7, 9**].

The golden-mean shift A *symbolic dynamical system* (X, σ) of the kind considered in this note consists of a set X of infinite sequences of symbols and a shift function σ that knocks off the first term of each sequence. As an example, let $\{0, 1\}$ be the symbol set, and let $X = \{0, 1\}^\infty$ be the set of all infinite 0-1 sequences of the form $x = c_0c_1c_2\cdots$, where $c_i = 0$ or 1. Define the shift function $\sigma : X \to X$ by $\sigma(x) = c_1c_2c_3\cdots$. For the open-tent function, we let G denote the subset of $\{0, 1\}^\infty$ that consists of the sequences in which adjacent zeros are forbidden. The set G together with the shift function σ defined above is called the *golden-mean shift*.

A directed graph H associated with G is shown in FIGURE 2. The vertices of H are the two symbols 0 and 1. The directed edges on H give the rule indicating which symbol can follow another in the sequences of G. Since there is no edge on H from 0 to itself, adjacent zeros are forbidden in the sequences of G. It is easy to see that the elements of G represent all infinite walks on H that start at either of the two vertices

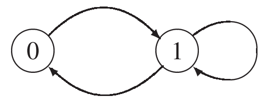

Figure 2 Directed graph H of golden-mean shift

and continue forever. The symbols in the sequence indicate the vertices visited during the walk in the order they are visited. The directed graph H can be recorded by the integer matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

as follows. Let the (i, j) entry, $A(i, j)$, of A be the number of edges from vertex i to vertex j. The matrix A is called the *adjacency matrix* of H. Note that the eigenvalues of A are the golden means

$$\frac{1 \pm \sqrt{5}}{2}.$$

Inductively, we have

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A^1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \ldots,$$
$$A^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix},$$

where $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, are the *Fibonacci numbers*, treated extensively elsewhere in this issue. It is well known that the nth power of an adjacency matrix counts the walks of length n on its graph. In fact, $A^n(i, j)$ counts the walks on H of length n from vertex i to vertex j, and the trace of A^n, $\operatorname{tr}(A^n) = A^n(1, 1) + A^n(2, 2)$, equals the number of closed walks of length n on H. The sequence $\{\operatorname{tr}(A^n)\}_{n=0}^{\infty}$ also satisfies the Fibonacci recurrence relation since $\operatorname{tr}(A^n) = F_{n-1} + F_{n+1}$ with $\operatorname{tr}(A^0) = 2 = L_0$ and $\operatorname{tr}(A^1) = 1 = L_1$. Therefore, $\{\operatorname{tr}(A^n)\}_{n=0}^{\infty}$ is the sequence of famous *Lucas numbers* L_n.

We use the notation $(c_0c_1 \cdots c_{n-1})^{\infty}$ to indicate the sequence in G or $\{0, 1\}^{\infty}$ formed by concatenating infinitely many copies of $c_0c_1 \cdots c_{n-1}$. Hence,

$$\sigma((c_0c_1 \cdots c_{n-1})^{\infty}) = (c_1c_2 \cdots c_{n-1}c_0)^{\infty} \quad \text{and}$$
$$\sigma^n((c_0c_1 \cdots c_{n-1})^{\infty}) = (c_0c_1 \cdots c_{n-1})^{\infty},$$

so $(c_0c_1 \cdots c_{n-1})^{\infty}$ has period n under σ. In G, then, we have only one fixed point $1^{\infty} = 11\cdots$ since $\sigma(1^{\infty}) = 1^{\infty}$ and 0^{∞} is not in G. We have two points $(01)^{\infty}$ and $(10)^{\infty}$ with prime period 2, since $\sigma((01)^{\infty}) = (10)^{\infty}$ and $\sigma((10)^{\infty}) = (01)^{\infty}$. The element $1^{\infty} = (11)^{\infty}$ also has period 2 though its prime period is 1.

The one-to-one correspondence between the elements of G and the infinite walks on H implies

$$\begin{pmatrix} \text{the number} \\ \text{of period-}n \\ \text{points in } G \end{pmatrix} = \begin{pmatrix} \text{the number of} \\ n\text{-step closed} \\ \text{walks in } H \end{pmatrix} = \operatorname{tr}(A^n) = L_n. \tag{1}$$

Encoding and decoding The link between a general dynamical system and its symbolic dynamical system is realized by the encoding and decoding processes. Through them we show that there is a one-to-one correspondence between the period n points of the open-tent function in [0, 1] and those of the golden-mean shift in G. We begin our encoding by making a partition of the unit interval, $I_0 = [0, 1/2)$ and $I_1 = [1/2, 1]$. We then encode every point $x \in [0, 1]$ as an infinite sequence as follows:

$$E(x) = c_0c_1c_2\cdots, \text{ where } c_k = 0 \text{ if } f^k(x) \in I_0 \text{ and } c_k = 1 \text{ if } f^k(x) \in I_1.$$

The sequence $E(x) = c_0c_1c_2\cdots$ is called the *encoding* of x (or the *itinerary* of x). It is the new ID of x. For the open-tent function f, we see that $f(I_0) \subseteq I_1$, so every 0 in the encoding of a number must be followed by a 1, that is, adjacent zeros are forbidden, and $E([0, 1]) \subseteq G$. As an example, since $f^0(0) = 0 \in I_0$, $f(0) = 1/2 \in I_1$, and $f^2(0) = 1 \in I_1$, the first three digits in the encoding of 0 are 011. Since f sends 1 back to 0, the sequence repeats. So, $E(0) = (011)^\infty$. Similarly, $E(1/2) = (110)^\infty$ and $E(1) = (101)^\infty$.

Though a brute force encoding is always possible, decoding is not as straightforward. To make a more general analysis possible, we move back and forth between x and $E(x)$ through the binary expansion of x. Things are complicated a bit by the fact that some rational numbers have two distinct binary expansions. For example, in binary $1/2 = 0.1\bar{0} = 0.0\bar{1}$ just as in decimal $1/2 = 0.5\bar{0} = 0.04\bar{9}$. We must proceed carefully.

Rational numbers of the form $j/2^n$ are called *dyadic numbers*. The dyadic numbers in (0, 1] are exactly the rational numbers in the unit interval that have two distinct binary expansions. If $x \in (0, 1]$ is dyadic, then there exist nonnegative integers j and n such that in lowest terms

$$x = \frac{j}{2^n} = 0.x_1x_2\ldots x_{n-1}1\bar{0} = 0.x_1x_2\ldots x_{n-1}0\bar{1}.$$

Before presenting technical coding formulas, let us see some heuristic descriptions. Because the second piece of f has a slope of -2, an interval in I_1 is stretched by f to double its length, and its orientation is reversed (if $x < y$, then $f(x) > f(y)$), while the first piece of f simply slides an interval in I_0 to the right 1/2 unit into I_1. The nth iteration of f is a piecewise function that is linear on dyadic intervals of the form $(p/2^n, (p+1)/2^n)$. For the dyadic numbers, we must make the proper choice of binary expansion. A function ψ is defined in (2) to serve this purpose.

Let us consider a generic case where $x \in [0, 1]$ is not dyadic. Suppose $x = 0.x_1x_2x_3\ldots$ is its binary expansion and $E(x) = c_0c_1c_2\cdots$ is its encoding. If $x_1 = 0$, then $x \in I_0$ and $f(x) \in I_1$, so we can determine that c_0c_1 must be 01. Similarly, if $x_1 = 1$, then we can determine that $c_0 = 1$. This is our first step of encoding through the binary expansion. Note that in the first case ($x_1 = 0$), f is applied once to determine the first two symbols in the encoding. In the second case ($x_1 = 1$), f is not applied at all and only the first symbol of the encoding was determined. In both cases, the encoding step ends with a symbol 1. That is, when the orbit enters I_1. To summarize:

$$x_1 = 0 \Rightarrow c_0c_1 = 01,$$
$$x_1 = 1 \Rightarrow c_0 = 1.$$

Having used the first digit of the binary expansion, we ignore it and focus on the second for step 2, because this digit determines the next entry in the encoding, whether it is c_1 or c_2. Suppose $x = 0.*x_2\ldots$. If $x_2 = 0$, then $x \in (0, 1/4)$ or $(1/2, 3/4)$. If the former, then we know that f moved $(0, 1/4)$ onto $(1/2, 3/4)$ without an orientation reversal in step 1. The next iteration of f sends $(1/2, 3/4)$ to I_1, so the next symbol in the coding is 1 and there is a total of one orientation reversal. Similarly, if $x_2 = 1$, then the next two symbols in the coding are 01 with one orientation reversal, and again, the step ends with the orbit entering I_1. Thus,

$$\begin{aligned} x_2 = 0 &\Rightarrow \text{ the next symbol in the encoding is } 1, \\ x_2 = 1 &\Rightarrow \text{ the next two symbols in the encoding are } 01. \end{aligned}$$

The third step deals with x_3. It produces another orientation reversal, so the orientation is the same as in step 1. Thus,

$$\begin{aligned} x_3 = 0 &\Rightarrow \text{ the next two symbols in the encoding are } 01, \\ x_3 = 1 &\Rightarrow \text{ the next symbol in the encoding is } 1. \end{aligned}$$

The process continues as above through the binary expansion of x. The encoding rules alternate, using x_2 as the template for x_n if n is even, and x_3 if n is odd. The argument, including the subtle handling of the dyadic numbers, is in the proof of Theorem 1. A casual reader could skip the proof.

We now develop technical algorithms for encoding and decoding. The expansions presented are binary. Define $\psi : [0, 1] \to \{0, 1\}^\infty$ by

$$\psi(x) = \begin{cases} x_1 x_2 \cdots & \text{if } x = 0.x_1 x_2 \ldots \text{ and is not dyadic, or } 0, \\ x_1 x_2 \cdots x_{n-1} 10^\infty & \text{if } x = 0.x_1 x_2 \ldots x_{n-1} 1\bar{0} \text{ and } n \text{ is odd,} \\ x_1 x_2 \cdots x_{n-1} 01^\infty & \text{if } x = 0.x_1 x_2 \ldots x_{n-1} 0\bar{1} \text{ and } n \text{ is even.} \end{cases} \tag{2}$$

We call $\psi(x)$ the *proper binary expansion* for x. Obviously, ψ is one-to-one and has a left inverse $\phi : \{0, 1\}^\infty \to [0, 1]$ defined by

$$\phi(z_1 z_2 z_3 \cdots) = 0.z_1 z_2 z_3 \ldots = \sum_{k=1}^{\infty} \frac{z_k}{2^k}$$

with $\phi \circ \psi = \mathrm{Id}_{[0,1]}$. It is also clear that ϕ is onto and almost one-to-one—except that it maps two binary expansion sequences to each nonzero dyadic number.

The next function, B, is a bijection between $\{0, 1\}^\infty$ and G. This function and its inverse are at the heart of the encoding and decoding processes, since they provide the correspondence between the proper binary expansion of a point and its encoding. Define $B : \{0, 1\}^\infty \to G$ by $B(z_1 z_2 z_3 \cdots) = y_1 y_2 y_3 \cdots$, where

$$y_n = \begin{cases} 01 & \text{if } n \text{ is odd and } z_n = 0, \\ 1 & \text{if } n \text{ is odd and } z_n = 1, \\ 1 & \text{if } n \text{ is even and } z_n = 0, \\ 01 & \text{if } n \text{ is even and } z_n = 1. \end{cases} \tag{3}$$

For example,

$$\begin{aligned} B(0^\infty) &= B(0000\cdots) = 011011\cdots = (011)^\infty \quad \text{and} \\ B(01^\infty) &= B(0111\cdots) = 0(101)^\infty. \end{aligned}$$

It is easy to see that B is a bijection with the inverse given by $B^{-1}(y_1 y_2 y_3 \cdots) = z_1 z_2 z_3 \cdots$ where $y_1 y_2 y_3 \cdots \in G$, y_n equals 01 or 1, and

$$z_n = \begin{cases} 0 & \text{if } n \text{ is odd and } y_n = 01, \\ 1 & \text{if } n \text{ is odd and } y_n = 1, \\ 1 & \text{if } n \text{ is even and } y_n = 01, \\ 0 & \text{if } n \text{ is even and } y_n = 1. \end{cases} \tag{4}$$

Define the *decoder* $D : G \to [0, 1]$ of E by $D = \phi \circ B^{-1}$. We have the following:

THEOREM 1. *$E = B \circ \psi$, $D \circ E = Id_{[0,1]}$, and $E \circ D|_{E([0,1])} = Id_{E([0,1])}$. In particular, the encoder E is one-to-one and the decoder D is onto.*

Proof. We first show that $E = B \circ \psi$. Suppose $x \in [0, 1]$ is not dyadic and $\psi(x) = x_1x_2x_3\cdots$. In addition, let $E(x) = c_0c_1c_2\cdots = y_1y_2y_3\cdots$, where c_k is 0 or 1 depending on whether $f^k(x)$ is in I_0 or I_1 and y_n is 01 or 1. Suppose further that $B \circ \psi(x) = y'_1y'_2y'_3\cdots$, where y'_n is 01 or 1. We show that $y_n = y'_n$ for all n by induction. If $x_1 = 0$, then $x = f^0(x) \in I_0$ and $f^1(x) \in I_1$, so $c_0 = 0$, $c_1 = 1$, and $y_1 = 01$. On the other hand, by the definition of ψ and (3), $x_1 = 0$ implies $y'_1 = 01$. Similarly, if $x_1 = 1$, then $x \in I_1$, so $y_1 = 1$ and $y'_1 = 1$. In either case, $y_1 = y'_1$.

Now assume $y_i = y'_i$ for $i = 1, \ldots, n-1$. Since x is not dyadic, there exists an integer j such that $0 \leq j \leq 2^{n-1} - 1$ and $x \in (j/2^{n-1}, (j+1)/2^{n-1})$, which lies entirely in I_0 or I_1. Each application of f sends such a dyadic interval to another dyadic interval. If the left branch of f is applied, its width remains the same, but if the right branch is applied, the width doubles and the endpoints of the image can both be written with the denominator 2^{n-2}. Such intervals still fall entirely in I_0 or I_1 until they are stretched to a width of 1. Therefore, upon the $(n-1)$st visit of x to I_1, this interval has been stretched to $(1/2, 1)$. If x is in the left half of $(j/2^{n-1}, (j+1)/2^{n-1})$, then $x_n = 0$. If n is even, then upon the $(n-1)$st visit to I_1 the left half of $(j/2^{n-1}, (j+1)/2^{n-1})$ has been stretched to $(1/2, 3/4)$. The next application of f sends the iteration of x already in $(1/2, 3/4)$ to I_1, so $y_n = 1$. But, n even and $x_n = 0$ implies $y'_n = 1$ by (3). Likewise, n odd implies $y_n = 01$ and $y'_n = 01$. In either case $y_n = y'_n$. The parallel argument shows that if x is in the right half of $(j/2^{n-1}, (j+1)/2^{n-1})$, then $x_n = 1$ and $y_n = y'_n$. Therefore, we proved that $y_n = y'_n$. for all n if x is not dyadic.

Suppose $x \in [0, 1]$ is dyadic. It is easy to check that $E(x) = B \circ \psi(x)$ for $x = 0,1/2$, or 1. If x is dyadic and different from those three, then $x = j/2^n$ in lowest terms with $n \geq 2$, and $x = j/2^n$ is the midpoint of an interval of the form $(p/2^{n-1}, (p+1)/2^{n-1})$ that falls entirely in I_0 or I_1. Let $E(x) = c_0c_1c_2\cdots = y_1y_2y_3\cdots$ as before, and suppose that $B \circ \psi(x) = y'_1y'_2y'_3\cdots$. Since the midpoint of $(p/2^{n-1}, (p+1)/2^{n-1})$ is an element of $(p/2^{n-1}, (p+1)/2^{n-1})$, an argument that parallels the nondyadic case shows that $y_k = y'_k$, but only for $k = 1, 2, \ldots n-1$. Upon the $(n-1)$st visit to I_1, the interval $(p/2^{n-1}, (p+1)/2^{n-1})$ is stretched onto $(1/2, 1)$ and $x = j/2^n$ is mapped to $3/4$. Let q equal the number of applications of f required to produce $n-1$ visits by $(p/2^{n-1}, (p+1)/2^{n-1})$ to I_1, then $c_0c_1c_2\cdots c_q = y_1y_2y_3\cdots y_{n-1}$ with $c_q = 1$. Another application of f sends x to $1/2$, so $c_{q+1} = 1$ and $y_n = 1$. Since $E(1/2) = (110)^\infty$, $E(x) = c_0c_1c_2\cdots c_{q-1}1(110)^\infty = y_1y_2y_3\cdots y_{n-1}(110)^\infty = y_1y_2y_3\cdots y_{n-1}1(101)^\infty$. Thus, if n is odd, $B \circ \psi(x) = B(x_1x_2x_3\cdots x_{n-1}10^\infty) = y'_1y'_2y'_3\cdots y'_{n-1}1(101)^\infty = E(x)$. Similarly, if n is even $B \circ \psi(x) = B(x_1x_2x_3\cdots x_{n-1}01^\infty) = y'_1y'_2y'_3\cdots y'_{n-1}1(101)^\infty = E(x)$.

By construction of the following maps

$$[0,1] \underset{\phi}{\overset{\psi}{\rightleftarrows}} \{0, 1\}^\infty \underset{B^{-1}}{\overset{B}{\rightleftarrows}} G,$$

we get $D \circ E = (\phi \circ B^{-1}) \circ (B \circ \psi) = \phi \circ \psi = \mathrm{Id}_{[0,1]}$. In particular, E is one-to-one and D is onto. Moreover, for any $y \in E([0, 1])$, there is an $x \in [0, 1]$ such that $E(x) = y$, so $E \circ D(y) = E(D(E(x))) = E(x) = y$, thus $E \circ D|_{E([0,1])} = \mathrm{Id}_{E([0,1])}$. This ends the proof. ■

Theorem 1 states that the encoder and the decoder are not inverses of each other, but almost. Hence, we cannot identify the open-tent function with its symbolic model the golden-mean shift. However, many important dynamical features like periodic points are still in one-to-one correspondence between the two systems.

Using (4) and $D(y) = \sum_{i=1}^{\infty} z_i/2^i$, we can decode any 0-1 sequence in G into a number in [0,1]. The first example is straightforward, but there are a few subtleties of decoding as demonstrated in Examples 2 & 3.

EXAMPLE 1. *To decode the element* $(11011)^\infty$*, use (4) directly to obtain* $D[(11011)^\infty] = 0.\overline{1000} = 2^3/(2^4 - 1) = 8/15$*. So,* $8/15$ *is a point of period* 5.

EXAMPLE 2. *A careless decoding may suggest that* $D[(1101)^\infty] = 0.\overline{100} = 4/7$*, but the odd number of ones in the string* 1101 *tells us that the even-odd parity is switched in the second appearance of* 1101 *in the infinite string* $(1101)^\infty$*. In this case we must list the repeating string twice to get an even number of* 1*s. Thus,* $D[(11011101)^\infty] = 0.\overline{100011} = 5/9$*, and* $5/9$ *has period* 4.

EXAMPLE 3. *What do we do with that final* 0 *when decoding* $(11110)^\infty$*? Simply note that* $(11110)^\infty = 1(11101)^\infty$*, so* $D[(11110)^\infty] = D[1(11101)^\infty] = 0.1\overline{0100} = 19/30$.

EXAMPLE 4. *What is the point* $x \in [0, 1]$ *such that* $f^n(x) \geq 1/2$ *for all n except when n equals one million? We get the answer by decoding an element of G with the right properties:*

$$x = D(1^{1,000,000}01^\infty) = \phi \circ B^{-1}[(11)^{500,000}(01)(11)^\infty] = \phi[(10)^{500,000}0(01)^\infty]$$
$$= \left(\sum_{k=0}^{\infty} \frac{1}{2^{2k+1}}\right) - \frac{1}{2^{10^6+1}} = \frac{2}{3} - \frac{1}{2^{10^6+1}} = \frac{2^{10^6+2} - 1}{3 \cdot 2^{10^6+1}}.$$

The encoder $E : [0, 1] \to G$ is not onto. Let us find exactly which elements of G fall outside $E([0, 1])$. Suppose $w \in G$, but $w \notin E([0, 1])$. Let $x = D(w) = \phi \circ B^{-1}(w)$. Both $B^{-1}(w)$ and $\psi(x)$ are binary expansion sequences of x by the definitions of ϕ and ψ respectively. But $B^{-1}(w) \neq \psi(x)$, for otherwise $w = B(B^{-1}(w)) = B(\psi(x)) = E(x) \in E([0, 1])$. A contradiction. Since ϕ is almost a bijection except for the dual representation of the dyadic numbers,

> *the elements of G that fall outside* $E([0, 1])$ *are precisely those elements of G that correspond through B to the improper binary expansions of the dyadic numbers.*

It is easy to spot these elements. If $y \in G$ begins with a 1, then $\sigma^{-1}(y) = \{0y, 1y\}$, while if y begins with a 0, then $\sigma^{-1}(y) = \{1y\}$. In FIGURE 3 we have an infinite directed graph for $(101)^\infty$ and its preimages. It shows the complete genealogy of the ambiguous sequences in G corresponding to the dyadic numbers. An arrow from w to y indicates $\sigma(w) = y$.

Since the dyadic numbers are the preimages of 1 under f^n for various n and $E(1) = (101)^\infty$, the elements of $E([0, 1])$ that are preimages of $(101)^\infty$ under σ for various n decode to the dyadic numbers. Notice, however, that $\sigma^{-1}[(101)^\infty] = \{(110)^\infty, 0(101)^\infty\}$ even though $f^{-1}(1) = \{1/2\}$. The element $0(101)^\infty \notin E([0, 1])$ and the whole right-hand branch of the directed graph in FIGURE 3 that passes through $0(101)^\infty$ lies outside of $E([0, 1])$.

> *The necessary and sufficient conditions for* $y \in G$ *being in* $E([0, 1])$ *are that either y does not have the repeating block* $(101)^\infty$*, or, if it does, then it has a* 1 *just before its repeating block* $(101)^\infty$.

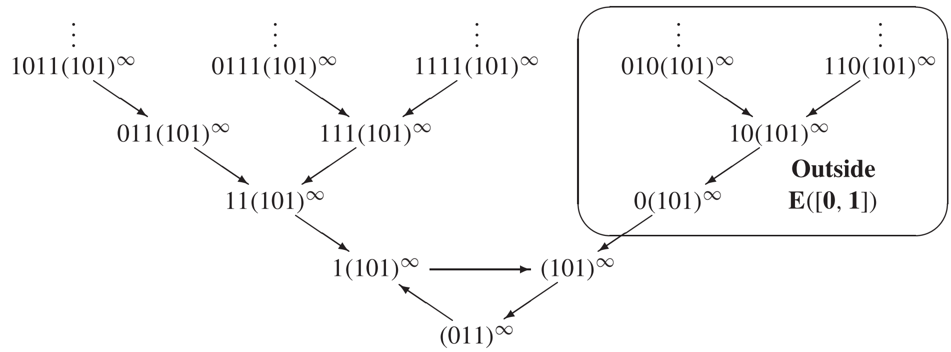

Figure 3 Preimages of $(101)^\infty$ under σ^n

None of the points in FIGURE 3 except the bottom three are periodic though all of their orbits eventually enter a periodic cycle. Such points are called *eventually periodic points*. Now we can describe all the periodic points and eventually periodic points of the open-tent function as our main result.

Periodic points and eventually periodic points

THEOREM 2. *Let f be the open-tent function on* $[0, 1]$*, and let* (G, σ) *be the golden-mean shift.*

1. *There is a period-preserving bijection between the set of all periodic points in* $([0, 1], f)$ *and those in* (G, σ)*. In particular, we can locate all periodic points of f precisely.*
2. *The number of period-n points of f equals the Lucas number L_n.*
3. *The only 3-cycle of f consists of the three dyadic numbers* $0 \xrightarrow{f} 1/2 \xrightarrow{f} 1 \xrightarrow{f} 0$*. All other dyadic numbers in* $[0, 1]$ *are eventually period-three points with their orbits eventually ending with the 3-cycle above. The converse is also true.*
4. *A number* $x \in [0, 1]$ *is a periodic point of f if and only if either*
   - *(a) x is a rational number that can be written as a fraction with an odd denominator (this includes 0 and 1), or*
   - *(b) x is a rational number that can be written in the form* $1/2 + j/(2k)$ *for some nonnegative integer j and odd positive integer k (this includes* $1/2$ *and* 1*).*
5. *A number* $x \in [0, 1]$ *is a periodic or eventually periodic point of f if and only if x is rational.*

Proof. (1,2). By design, if $E(x) = c_0c_1c_2\cdots$, then $E(f(x)) = c_1c_2c_3\cdots$. Hence, $E \circ f = \sigma \circ E$. This along with the fact that E is one-to-one (Theorem 1) implies that $f^n(x) = x$ if and only if $\sigma^n(E(x)) = E(x)$. Thus, x has period n under f if and only if $E(x)$ has period n under σ. Since none of the elements of G that fall outside $E([0, 1])$ are periodic (FIGURE 3), E serves as a bijection between the period-n points of $([0, 1], f)$ and those of (G, σ). So, the number of period-n points of $[0, 1]$ under f equals the Lucas number L_n by (1). To prove *(3)*, observe that the only prime-period-three elements in G are $(011)^\infty$, $(110)^\infty$, and $(101)^\infty$. They decode to the only prime-period-three elements 0, 1/2, and 1 respectively in $[0, 1]$. The proof of Theorem 1 implies the rest. We leave the proofs of *(4)* and *(5)* as exercises in the application of decoding. ■

TABLE 1 lists the period-n points of (G, σ) and $([0, 1], f)$ for $n = 1, 2, 3, 4$.

TABLE 1: Periodic points of the open-tent function

Prime Period	$E(x)$	x
1	1^∞	2/3
2	$(01)^\infty$, $(10)^\infty$	1/3, 5/6
3	$(011)^\infty$, $(110)^\infty$, $(101)^\infty$	0, 1/2, 1
4	$(0111)^\infty$, $(1110)^\infty$, $(1101)^\infty$, $(1011)^\infty$,	2/9, 13/18, 5/9, 8/9

Let q_n denote the number of points in [0, 1] having prime period n under f. If $k < n$ and k divides n, then the q_k elements of [0, 1] with prime period k are counted in L_n along with the q_n elements with prime period n. Thus,

$$q_n = L_n - \sum_{k|n, k<n} q_k.$$

With the help of a computer, we calculate some values of q_n in TABLE 2.

TABLE 2: Number of period-n points

n	No. of Period n Pts. L_n	No. of Prime Period n Pts. q_n
1	1	1
2	3	2
4	7	4
5	11	10
10	123	110
20	15,127	15,000
25	167,761	167,750
50	28,143,753,123	28,143,585,250
100	792,070,839,848,373,253,127	792,070,839,820,228,485,000

We should be aware of the limitations of a computer for such a seemingly simple process as calculating the iterations of f at some point x. Try using a spreadsheet or mathematical software that uses floating-point arithmetic to investigate this; you will find that *all orbits* of f end with the 3-cycle $0 \to 1/2 \to 1 \to 0$. Why? The computer uses a finite binary expansion to represent the seed number. In doing so, it has rounded the seed to a dyadic number. By Theorem 2, the orbits of all dyadic points end in that 3-cycle. This phenomenon is quite unique to the open-tent function. It is no longer true if we just move the top of the tent a bit higher or lower! Interested readers may study the orbit diagram (by *Maple*) in FIGURE 4 of the following family of functions with the parameter c:

$$f_c(x) = \begin{cases} cx + 1/2 & x < 1/2 \\ (1+c)(1-x) & x \geq 1/2 \end{cases}, \quad \text{for } -1 \leq c \leq \frac{1+\sqrt{5}}{2}.$$

The orbit diagram plots the parameter c with a gap of 0.02 on the horizontal axis versus the eventual orbit of the critical point 1/2 under f_c on the vertical axis. A different family that contains the open-tent function is discussed by Bassein [**2**].

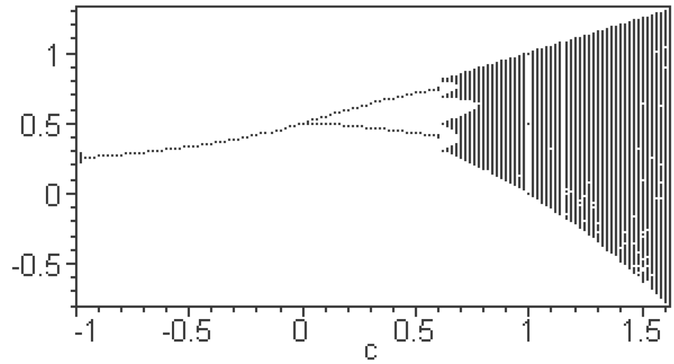

Figure 4 Orbit diagram of $x = 1/2$ under a modified tent function f_c

FIGURE 4 presents a familiar picture of transition to chaos through period-doubling bifurcations [**4**, Ch. 8]. For $-1 < c < 0$, the orbit of $x = 1/2$ tends to an attracting fixed point. At $c = 0$, the family has a period-doubling bifurcation where the attracting fixed point turns into a repelling fixed point and gives birth to an attracting 2-cycle. For $c > 0$, the orbit of $1/2$ tends to an attracting 2-cycle until $c \approx 0.617$ when another period-doubling bifurcation happens that gives birth to an attracting 4-cycle. The dark region of the diagram shows that the orbits of $x = 1/2$ under corresponding f_c are trapped in one or more vertical intervals, jumping back and forth chaotically. When $c = 1$, f_c is the open-tent function, and the orbit of $1/2$ is represented by the three dots that appear on the vertical line $c = 1$. The reason we can see these three dots is not because it is an attracting 3-cycle, but because all numbers are rounded by computer to dyadic numbers that eventually enter the 3-cycle $0 \to 1/2 \to 1 \to 0$ (see FIGURE 3). The open-tent function is unique in this family $\{f_c\}$. For c just off from 1, the orbit of $1/2$ under f_c is chaotic. When $c > (1 + \sqrt{5})/2$, the orbit of $1/2$ escapes, so we see the golden mean one more time to end the orbit diagram! We still do not know how to locate all the periodic points of f_c for all $c \neq 1$ as we do for the open-tent function.

Acknowledgment. The authors are grateful to the referees for valuable remarks and suggestions.

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Fibonacci Numbers and the Arctangent Function

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This note provides several geometric illustrations of three identities involving the arctangent function and the reciprocals of Fibonacci numbers. The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, for $n > 1$. The following identities link the Fibonacci numbers to the arctangent function. Only the first is evident in the literature [**1, 2, 3**].

$$\arctan\left(\frac{1}{F_{2i}}\right) = \arctan\left(\frac{1}{F_{2i+1}}\right) + \arctan\left(\frac{1}{F_{2i+2}}\right) \tag{1}$$

$$\arctan\left(\frac{2}{F_{2i+2}}\right) = \arctan\left(\frac{1}{F_{2i+1}}\right) + \arctan\left(\frac{1}{F_{2i+4}}\right) \tag{2}$$

$$\arctan\left(\frac{1}{F_{2i}}\right) = \arctan\left(\frac{2}{F_{2i+2}}\right) + \arctan\left(\frac{1}{F_{2i+3}}\right) \tag{3}$$

Identities (1)–(3) can be proven formally using Cassini's identity [**1**, p. 127]

$$F_{k+1}^2 = F_k F_{k+2} + (-1)^k$$

and the addition formula for the tangent function. Interested readers are invited to do so.

The following six diagrams illustrate special cases of equations (1)–(3). FIGURE 1, a representation of Euler's famous formula for π [**4, 5**], illustrates (1) for $i = 1$. One can see that $\angle ABD$ plus $\angle DBC$ is equal to $\angle ABC$.

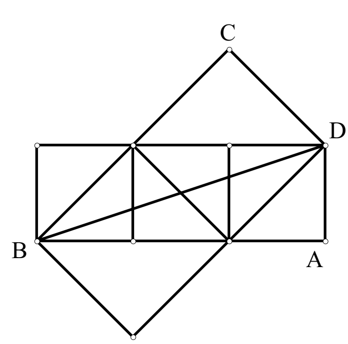

Figure 1 $\frac{\pi}{4} = \arctan(1) = \arctan(\frac{1}{2}) + \arctan(\frac{1}{3})$

FIGURE 2 illustrates (1) for $i = 2$, using the larger squares to form the arctangent of $1/5$ and the smaller squares being used to form the arctangents of $1/3$ and of $1/8$.

The two diagrams in FIGURE 3 illustrate (2) for the values $i = 1$ and $i = 2$.

The diagrams in FIGURE 4 illustrate equation (3) for the values $i = 1$ and $i = 2$.

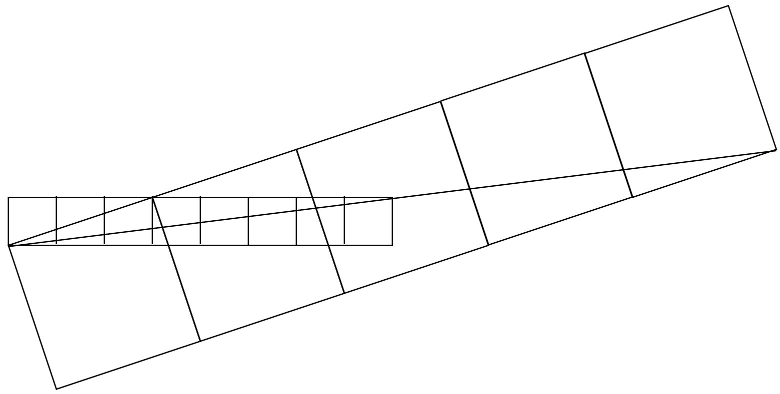

Figure 2 $\arctan(\frac{1}{3}) = \arctan(\frac{1}{5}) + \arctan(\frac{1}{8})$

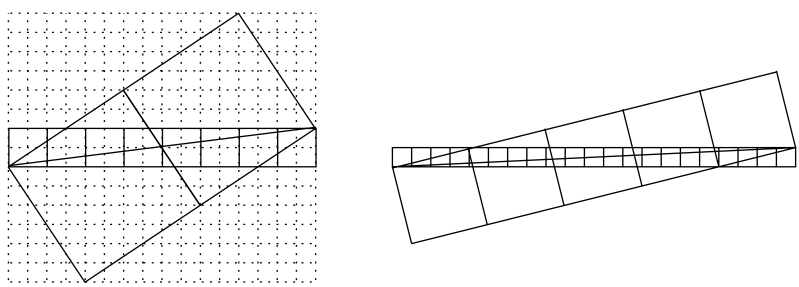

Figure 3 $\arctan(\frac{2}{3}) = \arctan(\frac{1}{2}) + \arctan(\frac{1}{8})$; $\arctan(\frac{1}{4}) = \arctan(\frac{1}{5}) + \arctan(\frac{1}{21})$

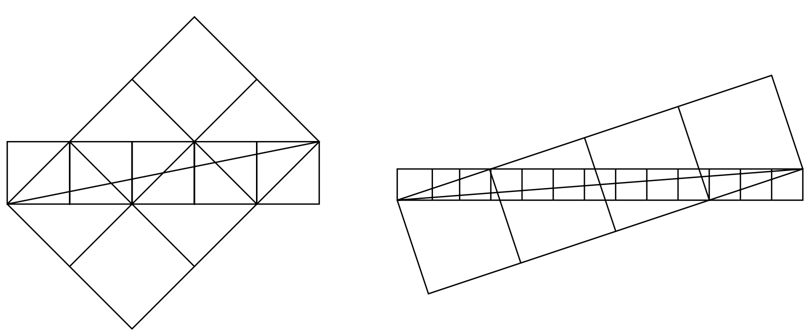

Figure 4 $\arctan(1) = \arctan(\frac{2}{3}) + \arctan(\frac{1}{5})$; $\arctan(\frac{1}{3}) = \arctan(\frac{1}{4}) + \arctan(\frac{1}{13}))$

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Hypercubes and Pascal's Triangle: A Tale of Two Proofs

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The entries of the nth row of Pascal's triangle consists of the combinatorial numbers

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-2}, \binom{n}{n-1}, \binom{n}{n}, \quad \text{where } \binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

These numbers are called the binomial coefficients, because they satisfy the binomial theorem:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k. \tag{1}$$

Upon setting $x = 1$, we obtain

$$2^n = \sum_{k=0}^{n} \binom{n}{k}. \tag{2}$$

Differentiating both sides of (1) with respect to x, we have

$$n(1+x)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} x^{k-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \cdots + n\binom{n}{n}x^{n-1}. \tag{3}$$

Setting $x = 1$, we finally obtain the well-known identity [**4**, p. 11],

$$n2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}. \tag{4}$$

This last identity can also be proven without calculus. For a typical short proof, see Rosen [**9**, Section 4.3, Exercise 51] or Buckley and Lewinter [**3**, Section 1.4, Exercise 9].

We shall prove identity (4) using graph theory. In contrast to the previously mentioned proofs, which suggest that (4) is an algebraic accident, our approach here will count a combinatorial object in two different ways, thereby yielding insight into *why* the identity is true. The hypercube, Q_n, is an important graph, with applications in computer science [**1**]–[**3**], [**5**]–[**8**]. Its vertex set is given by $V(Q_n) = \{(x_1, x_2, \ldots, x_n) \mid x_i = 0 \text{ or } 1; i = 1, 2, \ldots, n\}$, i.e., each vertex is labeled by a binary n-dimensional vector. It follows that $|V(Q_n)| = 2^n$. Vertices $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are adjacent if and only if $\sum_{i=1}^{n} |x_i - y_i| = 1$, from which it follows that Q_n is n-regular. Since the degree sum is $n2^n$, we find that Q_n has $n2^{n-1}$

edges, that is, $|E(Q_n)| = n2^{n-1}$. The distance between vertices x and y is given by $\sum_{i=1}^{n} |x_i - y_i|$, that is, the number of place disagreements in their binary vectors.

Calling the vertex $(0, 0, \dots, 0)$ the *origin*, define the ith *distance set* D_i, as the set of vertices whose distance from the origin is i. Then for each $i = 0, 1, 2, \dots, n$, we have $D_i = \{(x_1, x_2, \dots, x_n) \mid \sum_{i=1}^{n} x_i = i\}$, that is, D_i consists of those vertices with exactly i 1s in their binary n-vectors. Moreover, we have $|D_i| = \binom{n}{i}$. The fact that the D_is partition $V(Q_n)$ demonstrates Equation (2) rather nicely.

Now observe that the induced subgraph on any D_i contains no edges, since all of the binary vectors of the vertices in D_i contain the same number of 1s. (If two vertices are adjacent, the number of 1s in their binary vectors must differ by exactly one.) Furthermore, if $|i - j| \geq 2$, then if $x \in D_i$ and $y \in D_j$, it follows that x and y are nonadjacent, that is, $xy \notin E(Q_n)$. Then all edges are of the form uv, where $u \in D_i$ and $v \in D_{i+1}$, for $i = 0, 1, 2, \dots, n-1$. Since each vertex in D_{i+1} has $i+1$ 1s in its binary vector, it is adjacent to exactly $i+1$ vertices in D_i. (These vertices are obtained by replacing one 1 by 0 in the binary vector of the chosen vertex in D_{i+1}.) This implies that the number of edges with endpoints in both D_i and D_{i+1} is $(i+1)|D_{i+1}| = (i+1)\binom{n}{i+1}$. It follows that the total number of edges in Q_n is given by $\sum_{i=0}^{n-1} (i+1)\binom{n}{i+1}$. Finally, since $|E(Q_n)| = n2^{n-1}$, we are done with the proof of (4).

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A Derivation of Taylor's Formula with Integral Remainder

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Taylor's formula with integral remainder is usually derived using integration by parts [**4, 5**], or sometimes by differentiating with respect to a parameter [**1, 2**]. According to M. Spivak [**7**, p. 390], integration by parts is applied in a "rather tricky way" to derive Taylor's formula, using a substitution that "one might discover after sufficiently many similar but futile manipulations". In this MAGAZINE, Lampret [**3**] derived both Taylor's formula and the Euler-Maclaurin summation formula using a rather heroic application of integration by parts.

We derive the remainder formula in a way that avoids tricks and heroics. The key step is changing the order of integration in multiple integrals, a topic that many students in an analysis class will benefit from reviewing. This derivation has almost certainly been found many times before [**6**], however, most people seem to be unaware of it.

The Taylor formula Suppose that a function $f(x)$ and all its derivatives up to $n+1$ are continuous on the real line. Then Taylor's formula for $f(x)$ about 0 is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + R(x), \tag{1}$$

where the remainder, $R(x)$, is given by

$$R(x) = \frac{1}{n!}\int_0^x (x-u)^n f^{(n+1)}(u)\, du.$$

Our derivation is based on the following simple idea: Try to reconstruct f by integrating $f^{(n+1)}$, $n+1$ times. This approach is suggested by the case $n=0$, when (1) is merely the fundamental theorem of calculus. For notational simplicity, we prove (1) for only $n=2$; however, the general case is similar. Thus, consider

$$\widetilde{R}(x) := \int_0^x \int_0^w \int_0^v f^{(3)}(u)\, du\, dv\, dw. \tag{2}$$

Now let's evaluate this integral in two ways. The first way is by direct integration using the fundamental theorem of calculus three times:

$$\widetilde{R}(x) = f(x) - f(0) - xf'(0) - \frac{x^2}{2!}f''(0). \tag{3}$$

The second way to integrate (2) is by interchanging the order of integration:

$$\int_0^w \int_0^v f^{(3)}(u)\, du\, dv = \int_0^w \int_u^w f^{(3)}(u)\, dv\, du = \int_0^w (w-u) f^{(3)}(u)\, du.$$

Interchanging the order of integration again gives

$$\begin{aligned}\int_0^x \left\{\int_0^w \int_0^v f^{(3)}(u)\, du\, dv\right\} dw &= \int_0^x \left\{\int_0^w (w-u) f^{(3)}(u)\, du\right\} dw \\ &= \int_0^x \int_u^x (w-u) f^{(3)}(u)\, dw\, du \\ &= \frac{1}{2}\int_0^x (x-u)^2 f^{(3)}(u)\, du. \end{aligned} \tag{4}$$

Equating (3) and (4) yields the Taylor formula (1) for $n=2$.

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A Theorem Involving the Denominators of Bernoulli Numbers

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The Swiss mathematician, Jakob Bernoulli (1654–1705), successfully sought a general method for summing the first n kth powers for arbitrary positive integers n and k. Let us define

$$S_k(n) = \sum_{j=1}^{n} j^k = 1^k + 2^k + \cdots + n^k.$$

Define the average of the first n kth powers by

$$\mu_k(n) = \frac{S_k(n)}{n}.$$

We pose and answer the following natural question: For which values of n and k is $\mu_k(n)$ an integer? Our answer, although it does involve the denominators of Bernoulli numbers, which undergraduates may not have seen, relies primarily upon elementary divisibility arguments.

Background In his *Ars Conjectandi*, published posthumously in 1713 and dedicated primarily to the theory of probability, Bernoulli presented a recursive solution for $S_k(n)$. It states that for $k \geq 1$,

$$(n+1)^{k+1} = (n+1) + \sum_{j=1}^{k} \binom{k+1}{j} S_j(n),$$

where the binomial coefficients are defined as usual:

$$\binom{k+1}{j} = \frac{(k+1)!}{j!\,(k+1-j)!}.$$

Furthermore, if we define what are now called the *Bernoulli numbers* by

$$B_0 = 1 \quad \text{and} \quad (k+1)B_k = -\sum_{j=0}^{k-1} \binom{k+1}{j} B_j \quad \text{for } k \geq 1,$$

then for $k \geq 1$, the sums $S_k(n)$ satisfy:

$$(k+1)S_k(n) = \sum_{j=0}^{k} \binom{k+1}{j} B_j (n+1)^{k+1-j}.$$

The Bernoulli numbers are the rational coefficients of the linear terms of the $(k+1)$st degree polynomials, $S_k(n-1)$. For example,

$$\begin{aligned} S_0(n-1) &= 1n - 1, \\ S_1(n-1) &= \frac{1}{2}n^2 - \frac{1}{2}n, \\ S_2(n-1) &= \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n, \\ S_3(n-1) &= \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2 + 0n, \quad \text{and} \\ S_4(n-1) &= \frac{1}{5}n^5 - \frac{1}{2}n^4 - \frac{4}{15}n^3 - \frac{1}{30}n. \end{aligned}$$

It follows that $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, and $B_4 = -1/30$. In fact, $B_{2k+1} = 0$ for all $k \geq 1$. More compactly, we can define the Bernoulli numbers by the following power series:

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!}.$$

For even $k \geq 2$, we write $B_k = N_k/D_k$, where N_k and D_k are relatively prime and $D_k \geq 1$. The numerators N_k have played a significant role in number theory due largely to their connection with Fermat's Last Theorem. A prime p is a *regular* prime if p does not divide any of the numbers $N_2, N_4, \ldots, N_{p-3}$. (The only irregular primes less than 100 are 37, 59, and 67.) In 1850, Ernst Kummer proved that Fermat's Last Theorem is true for every exponent that is a regular prime. Of course, as of 1995 Andrew Wiles has proved Fermat's Last Theorem in toto.

The denominators D_k have played a less significant role in mathematics even though they can be clearly described. The 1840 Von Staudt-Clausen Theorem states that for k even, D_k is the product of all primes p with $(p-1) \mid k$. An interesting consequence is that D_k is square-free for all k. The theorem was proven independently (and nearly simultaneously) by the two mathematicians.

Examples We begin by considering a few examples, deriving results directly using congruence relations.

- $k = 1$: We have $\mu_1(n) = (n+1)/2$. Hence, $\mu_1(n) \in \mathbb{Z}$ if and only if n is odd. This is an exceptional case due to the fact that $B_1 \neq 0$.
- $k = 2$: In this case, $\mu_2(n) = (n+1)(2n+1)/6$. We claim that $\mu_2(n) \in \mathbb{Z}$ if and only if n is not divisible by 2 or 3. First, suppose that n is not divisible by 2

or 3. Clearly, $(n+1)(2n+1)$ is even. If $n \equiv 1 \pmod 3$, then $3 \mid (2n+1)$ and if $n \equiv 2 \pmod 3$, then $3 \mid (n+1)$. In any event, $6 \mid (n+1)(2n+1)$ and so $\mu_2(n) \in \mathbb{Z}$. Second, suppose that n is divisible by either 2 or 3. If $2 \mid n$, then $(n+1)(2n+1)$ is odd and hence is not divisible by 6. If $3 \mid n$, then $n = 3k$ for appropriate integer k, and $(n+1)(2n+1) = (3k+1)(6k+1) = 18k^2 + 9k + 1$, a number not divisible by 3 (nor by 6).

- $k = 3$: We have $\mu_3(n) = n(n+1)^2/4$. We claim that $\mu_3(n) \in \mathbb{Z}$ as long as n is not congruent to 2 modulo 4. If n is congruent to 0, 1, or 3 modulo 4, then $4 \mid n(n+1)^2$. However, if $n \equiv 2 \pmod 4$, then $n(n+1)^2 \equiv 2 \pmod 4$, and so 4 does not divide $n(n+1)^2$.
- $k = 4$: In this case, $\mu_4(n) = (n+1)(2n+1)(3n^2+3n-1)/30$. We claim that $\mu_4(n) \in \mathbb{Z}$ if and only if n is not divisible by 2, 3, or 5. Suppose that n is relatively prime to 30 (equivalently not divisible by 2, 3, or 5). Then $n+1$ is even and $(n+1)(2n+1)$ is divisible by 3. Furthermore, $(n+1)(2n+1)(3n^2+3n-1) = 6n^4 + 15n^3 + 10n^2 - 1 \equiv n^4 - 1 \pmod 5$. But by Fermat's Little Theorem, $n^4 - 1 \equiv 0 \pmod 5$ and so $5 \mid (n+1)(2n+1)(3n^2+3n-1)$. Hence, $\mu_4(n) \in \mathbb{Z}$ in this case as well.

  In the other direction, if $2 \mid n$, then $(n+1)(2n+1)(3n^2+3n-1)$ is odd and not divisible by 30. If $3 \mid n$, then $(n+1)(2n+1)(3n^2+3n-1) \equiv -1 \pmod 3$ and so is not divisible by 30. Finally, if $5 \mid n$, then $(n+1)(2n+1)(3n^2+3n-1) \equiv -1 \pmod 5$ and so is not divisible by 30.

These examples hint that the situation is very different for odd and even values of n. We develop our main theorem in two sections. Only the even case involves the Bernoulli numbers. In both parts, we use the easily noted fact that $\mu_k(n)$ is an integer if and only if $S_k(n) \equiv 0 \pmod n$.

An "odd" theorem

THEOREM 1. *For odd numbers $k \geq 3$, $\mu_k(n)$ is an integer if and only if $n \not\equiv 2 \pmod 4$.*

Proof. Suppose k is odd and $k \geq 3$. Since $(n-a)^k \equiv -a^k \pmod n$ for all a, we can pair up the terms of $S_k(n)$ from the outside in.

(a) If n is odd, then

$$\begin{aligned} S_k(n) &= [1^k + (n-1)^k] + [2^k + (n-2)^k] + \cdots \\ &\quad + \left[\left(\frac{n-1}{2}\right)^k + \left(\frac{n+1}{2}\right)^k\right] + n^k \\ &\equiv (1^k - 1^k) + (2^k - 2^k) + \cdots + 0 = 0 \pmod n. \end{aligned}$$

(b) If n is even, then there are two subcases depending on whether or not n is divisible by 4.

(i) If $n \equiv 0 \pmod 4$, then

$$\begin{aligned} S_k(n) &= [1^k + (n-1)^k] + [2^k + (n-2)^k] + \cdots \\ &\quad + \left[\left(\frac{n}{2}-1\right)^k + \left(\frac{n}{2}+1\right)^k\right] + \left(\frac{n}{2}\right)^k + n^k \\ &\equiv 0 \pmod n \quad \text{since } k > 1 \text{ and } \frac{n}{2} \text{ is even.} \end{aligned}$$

(ii) If $n \equiv 2 \pmod 4$, then

$$\begin{aligned} S_k(n) &= [1^k + (n-1)^k] + [2^k + (n-2)^k] + \cdots \\ &\quad + \left[\left(\frac{n}{2}-1\right)^k + \left(\frac{n}{2}+1\right)^k\right] + \left(\frac{n}{2}\right)^k + n^k \\ &\equiv \left(\frac{n}{2}\right)^k \pmod n. \end{aligned}$$

But $n/2$ is odd and so $(n/2)^k$ is odd. Since n is even, $(n/2)^k$ is not congruent to $0 \pmod n$. ■

An "even" more interesting theorem

THEOREM 2. *For even numbers $k \geq 2$, $\mu_k(n)$ is an integer if and only if n is relatively prime to D_k.*

Proof. The Von Staudt-Clausen theorem [**1**, Theorem 118] states that the kth Bernoulli denominator can be written as a product of primes as follows:

$$D_k = \prod_{p \text{ prime and } p-1|k} p.$$

To prove our result it must be shown that $S_k(n) \equiv 0 \pmod n$ if and only if for every prime p dividing n, that $p \nmid D_k$. By Von Staudt-Clausen it suffices to establish that

$$S_k(n) \equiv 0 \pmod n \text{ iff for every prime } p \text{ that divides } n,\ (p-1) \nmid k. \tag{1}$$

For the sake of completeness, we state and prove the following easily established result [**1**, Theorem 119]:

LEMMA 1. *For any prime p,*

$$\begin{aligned} \sum_{m=1}^{p} m^k &\equiv -1 \pmod p \text{ if } (p-1) \mid k \\ &\equiv 0 \pmod p \text{ if } (p-1) \nmid k. \end{aligned} \tag{2}$$

Proof of Lemma 1. If $(p-1) \mid k$, then $k = (p-1)r$ for some integer r. Hence, for $m < p$, $m^k = (m^{p-1})^r \equiv 1 \pmod p$ by Fermat's Little Theorem. It follows that $\sum_{m=1}^{p} m^k \equiv p - 1 \equiv -1 \pmod p$.

If $(p-1) \nmid k$, then let g be a primitive root of p. It follows that the set $\{g, 2g, \ldots, (p-1)g\}$ is identical to the set $\{1, 2, \ldots, p-1\}$ of reduced residues modulo p. Hence $\sum_{m=1}^{p-1} (mg)^k \equiv \sum_{m=1}^{p-1} m^k \pmod p$, and so $(g^k - 1) \sum_{m=1}^{p-1} m^k \equiv 0 \pmod p$.

But g^k is not congruent to $1 \pmod p$ since g is a primitive root mod p. Thus $\sum_{m=1}^{p} m^k \equiv 0 \pmod p$. This establishes Lemma 1. ■

Returning to the proof of our main result, it is convenient to first assume that n *is square-free*.

We establish (1):

($\Leftarrow$) Suppose that for all p dividing n that $(p-1) \nmid k$. Choose a prime $p \mid n$. By (2),

$$\sum_{m=1}^{p} m^k \equiv 0 \pmod p.$$

Similarly,

$$\sum_{m=rp+1}^{(r+1)p} m^k \equiv 0 (\text{mod } p) \quad \text{for } 0 \le r \le \frac{n}{p} - 1.$$

Hence $S_k(n) = \sum_{m=1}^{n} m^k = \sum_{r=0}^{\frac{n}{p}-1} \sum_{m=rp+1}^{(r+1)p} m^k \equiv 0(\text{mod } p)$. But p arbitrary and n square-free implies that $S_k(n) \equiv 0(\text{mod } n)$.

(⇒) We prove the contrapositive. Suppose there exists a prime $p \mid n$ such that $(p-1) \mid k$. By (2)

$$\sum_{m=1}^{p} m^k \equiv -1(\text{mod } p).$$

Similarly,

$$\sum_{m=rp+1}^{(r+1)p} m^k \equiv -1(\text{mod } p) \quad \text{for } 0 \le r \le \frac{n}{p} - 1.$$

Hence $S_k(n) \equiv -n/p(\text{mod } p)$, which is not congruent to $0(\text{mod } p)$ since p and n/p are relatively prime. Thus $S_k(n)$ is not congruent to $0(\text{mod } n)$.

Now suppose that n *is not square-free*.

(⇐) Suppose that for all p dividing n that $(p-1) \nmid k$. If there is a prime p exactly dividing n (that is, $p \mid n$, but p^2 does not divide n), then as in the square-free case, $S_k(n) \equiv 0(\text{mod } p)$.

Now let p be a prime with $p^a \mid\mid n$ with $a \ge 2$. (The notation $p^a \mid\mid n$ means that $p^a \mid n$ and $p^{a+1} \nmid n$.)

LEMMA 2. *Let p be a prime with $(p-1) \nmid k$. Then*

$$1^k + 2^k + \cdots + (p^a)^k \equiv 0(\text{mod } p^a).$$

Proof of Lemma 2 (Induction on a). If $a = 1$, then

$$1^k + 2^k + \cdots + p^k \equiv 0(\text{mod } p) \text{ by (2).}$$

Assume that the lemma holds for $a-1$, namely that

$$1^k + 2^k + \cdots + (p^{a-1})^k \equiv 0(\text{mod } p^{a-1}).$$

Now consider $S_k(p^a) = \sum_{r=0}^{p-1} \sum_{j=1}^{p^{a-1}} (rp^{a-1} + j)^k$. The binomial theorem implies that

$$(rp^{a-1} + j)^k = \sum_{i=0}^{k} \binom{k}{i} r^i p^{(a-1)i} j^{k-i}.$$

Hence

$$S_k(p^a) = \sum_{r=0}^{p-1} \sum_{j=1}^{p^{a-1}} \sum_{i=0}^{k} \binom{k}{i} r^i p^{(a-1)i} j^{k-i}. \tag{3}$$

For $i \ge 2$, $p^{(a-1)i} \equiv 0(\text{mod } p^a)$ and so all terms of (3) with $i \ge 2$ are congruent to $0(\text{mod } p^a)$.

For $i = 0$, $\sum_{r=0}^{p-1} \sum_{j=1}^{p^{a-1}} j^k = p \cdot S_k(p^{a-1})$.
But $S_k(p^{a-1}) \equiv 0 (\bmod p^{a-1})$ by our inductive hypothesis. Hence

$$\sum_{r=0}^{p-1} \sum_{j=1}^{p^{a-1}} j^k \equiv 0 (\bmod p^a).$$

For $i = 1$,

$$\sum_{r=0}^{p-1} \sum_{j=1}^{p^{a-1}} krp^{a-1} j^{k-1} = \sum_{r=0}^{p-1} krp^{a-1} \cdot S_{k-1}(p^{a-1})$$

$$= kS_{k-1}(p^{a-1}) \cdot p^{a-1} \cdot \frac{(p-1)p}{2}.$$

But $S_{k-1}(p^{a-1}) \in \mathbb{Z}$ and $2 \mid (p-1)$. Thus

$$\sum_{r=0}^{p-1} \sum_{j=1}^{p^{a-1}} krp^{a-1} j^{k-1} \equiv 0 (\bmod p^a).$$

Therefore, $S_k(p^a) \equiv 0 (\bmod p^a)$ and Lemma 2 is proven. ■

In a manner analogous to Lemma 2, it follows that

$$\sum_{m=rp^a+1}^{(r+1)p^a} m^k \equiv 0 (\bmod p^a) \quad \text{for } 0 \le r \le \frac{n}{p^a} - 1.$$

Hence $S_k(n) \equiv 0 (\bmod p^a)$ for any $p \mid n$ with $p^a \mid\mid n$ and $a \ge 1$. It follows that $S_k(n) \equiv 0 (\bmod n)$.

($\Rightarrow$) A slight modification of the square-free proof works here, as follows.

On the one hand, if there exists a prime $p \mid\mid n$ such that $(p-1) \mid k$, then by (2), $\sum_{m=1}^{p} m^k \equiv -1 (\bmod p)$. Hence $S_k(n) \equiv -n/p (\bmod p)$, which is not congruent to $0 (\bmod p)$ since $p \mid\mid n$. Thus $n \nmid S_k(n)$ as in the square-free case.

On the other hand, suppose there exists a prime p with $p^a \mid\mid n$ with $a \ge 2$ and $(p-1) \mid k$. By (2), $\sum_{m=1}^{p} m^k \equiv -1 (\bmod p)$. Thus $\sum_{m=1}^{p} m^k \equiv (rp-1)$ $(\bmod p^a)$ for some r with $1 \le r \le p^{a-1}$. But then $S_k(n) \equiv n/p(rp-1) \equiv -n/p (\bmod p^a)$. Hence $S_k(n)$ is not congruent to $0 (\bmod p^a)$ and so $n \nmid S_k(n)$.

This completes the proof of part (2) and establishes the theorem. ■

REFERENCES

1. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th ed., Oxford, Clarendon Press, 1979.

Proof Without Words: Viviani's Theorem with Vectors

The sum of the distances from a point P in an equilateral to the three sides of the triangle is independent of the position of P (and so equal to the altitude of the triangle.)

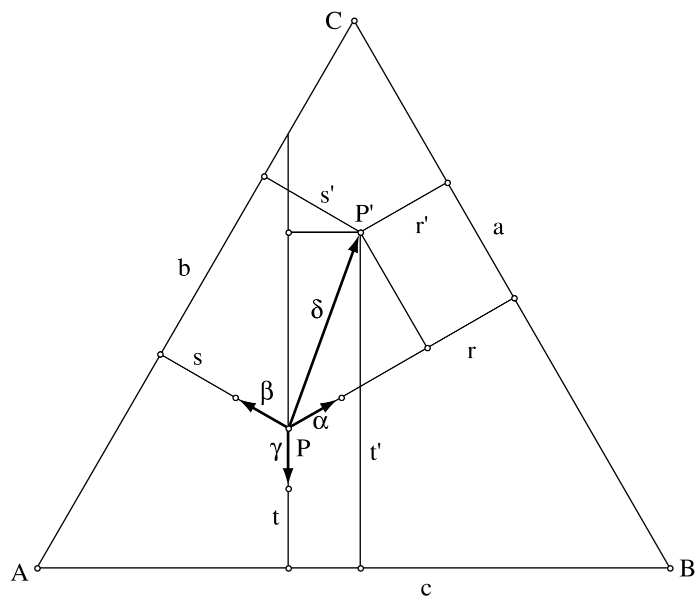
$$|\alpha| = |\beta| = |\gamma|$$

$$\alpha + \beta + \gamma = 0$$

$$\alpha \cdot \delta + \beta \cdot \delta + \gamma \cdot \delta = 0$$

$$(r - r') + (s - s') + (t - t') = 0$$

$$r + s + t = r' + s' + t'$$



—HANS SAMELSON
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Let π be 3

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And he (Hirum of Tyre) made a molten sea, ten cubits from the one brim to the other: it was round all about, and its height was five cubits: and a line of thirty cubits did compass it round about. 1 Kings 7:23

The literal interpretation of this Biblical passage is that Hirum constructed a hemispherical basin that had a diameter of 10 cubits (a cubit is approximately one half of a meter) and had a circumference three times that value. This ratio of the circumference of a circle to its diameter apparently contradicts results of the works of Archimedes who established that the ratio of the circumference of any circle to its diameter is between $3\frac{10}{71}$ and $3\frac{1}{7}$. Since Archimedes provided a convincing argument for his values we are inclined to accept them as true and regard the numbers given to us in First Kings as approximate values whose error is due to rounding off. In this paper we do not write this off to a rounding error, but rather identify a setting where 3 is the correct value.

Archimedes' results were obtained in Euclidean geometry. Using alternate geometries we will establish that the ratio of the circumference of a circle to its diameter may take on a continuum of values, including three. In the next section we will discuss how Archimedes first determined his values. Then we will show how the ratio varies in spherical geometry. Finally, we discuss the possible values in hyperbolic geometry.

The results of Archimedes The computation of π has a long history [**2**]. Archimedes [**1**] first considered a regular polygon as inscribed within a circle and then as circumscribed about a circle, and thus was able to compute a lower approximation and an upper approximation for the ratio of the circumference of a circle to its diameter. He observed these ratios up to a polygon with 96 sides, and thus was able to conclude that the ratio of the circumference of any circle to its diameter is between $3\frac{10}{71}$ and $3\frac{1}{7}$. Using Archimedes' technique and modern trigonometry we can compute even better approximations for this ratio.

Let us assume that a circle is divided into 360 degrees, and consider regular polygons having n sides (in our diagrams $n = 6$), where the length of each of the sides is 1 unit.

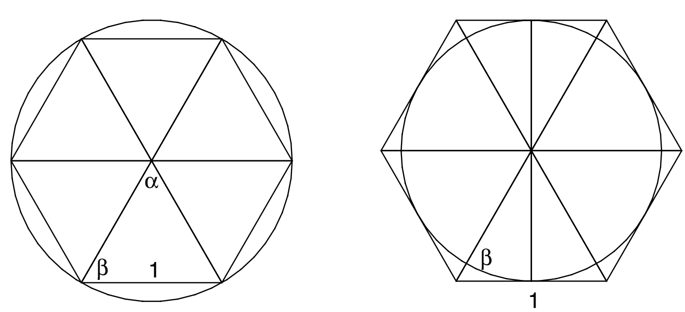

Figure 1 A circle circumscribed and inscribed by a regular polygon

From the easy observation that the measure of the angle α is $360°/n$, we find that the measure of the angle β is $90°(n-2)/n$. Now using the law of sines, we can see that the diameter of the circumscribed circle would be

$$2 \cdot \frac{\sin(90° \cdot (n-2)/n)}{\sin(360°/n)}.$$

The diameter of the inscribed circle is easily calculated to be

$$\tan(90° \cdot (n-2)/n).$$

We see that the ratio of the circumference of the circle to its diameter lies between ratio of the perimeter of the inscribed polygon to the circle's diameter and the ratio of the perimeter of the circumscribed polygon to the circle's diameter. For the remainder of the paper we will follow the universally accepted convention of using π as the ratio of the circumference of a circle to its diameter in Euclidean geometry. Computing the ratio of the perimeter of each polygon to the diameter of the circle, we derive the inequality

$$n \cdot \frac{\sin(360^\circ/n)}{2 \cdot \sin(90^\circ(n-2)/n)} < \pi < n \cdot \cot(90^\circ(n-2)/n).$$

We can now take the limits as n approaches infinity. The results yield what we already know, that each ratio approaches the same value, which is approximately 3.141592654.

The ratio π in spherical geometry From First Kings the ratio of the circumference to the diameter is easily computed to be 3. For the sake of academic curiosity let us assume that the ratio of the circumference to the diameter of the "molten sea" was indeed supposed to be 3. How can this be? An answer lies in the geometry under consideration. Archimedes proved his results in Euclidean geometry, the geometry of plane or flat surfaces. But suppose we were to use spherical geometry. Can we then produce circles in which the ratios of the circumferences to the diameters are indeed exactly 3?

In Euclidean geometry, the surface upon which measurements are made is the plane. In spherical geometry, the universal surface upon which measurements are made is the sphere. To perform calculations on the sphere we imagine the sphere embedded in three-dimensional Euclidean space. We may then use the results of Euclidean geometry to compute measurements on the sphere.

In the spherical setting, the ratio of circumference to diameter need not be π; in fact, it is easy to see how to produce a circle whose circumference-to-diameter ratio is 2. Simply choose the circle to be a great arc (equator), then the diameter of the circle (the shortest path between two antipodal points on the circle) is half the length of the great arc. So the ratio of the great arc to 1/2 of a great arc is $1/(1/2) = 2$.

Since we now know that on a sphere the ratio of a circumference of a circle to its diameter need not be π, we turn our attention to all possible values for this ratio.

Let a circle with circumference C and diameter $2r$ lie on a sphere with radius R. Let α be the center of the circle, β the center of the sphere, γ a point on the circle, and δ the center of the circle as viewed in Euclidean geometry. Call ρ the Euclidean radius of the circle in the planar cross-section shown, and θ the angle $\angle\alpha\beta\gamma$, as in FIGURE 2.

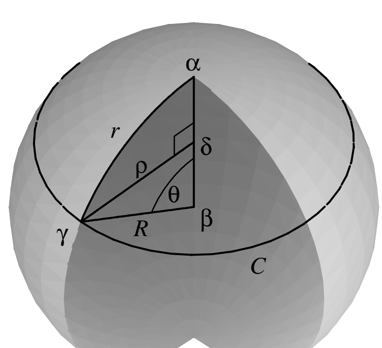

Figure 2 A circle on a sphere

We have $r = R\theta$, $\sin\theta = \rho/R$, and $2\rho\pi = C$, so that $\rho = R\sin\theta$ and $\theta = r/R$. Thus

$$C = 2\pi R \sin\frac{r}{R} = 2\pi\frac{r}{\theta}\sin\theta.$$

Let Π denote the ratio of the circumference of a circle to its diameter, which we represent as a function of θ. Since $2r$ is the diameter of the circle on the sphere, we can compute

$$\Pi(\theta) = \frac{C(\theta)}{2r} = \pi\frac{\sin\theta}{\theta}.$$

We now can compute $\Pi(\pi/2) = 2$, as we noted earlier. We can also compute $\Pi(\pi/6) = 3$, which is the ratio given in First Kings. We also note that the limiting value as the angle θ approaches 0 is

$$\lim_{\theta\to 0}\Pi(\theta) = \pi\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \pi.$$

We may allow θ to grow larger than $\pi/2$ and define the diameter of the circle to be twice the radius, which is the length of the arc from a point on the circle to the center, which we might as well call the North Pole. The graph of $\Pi(\theta) = \pi\sin(\theta)/\theta$ will represent all possible values for Π on the sphere.

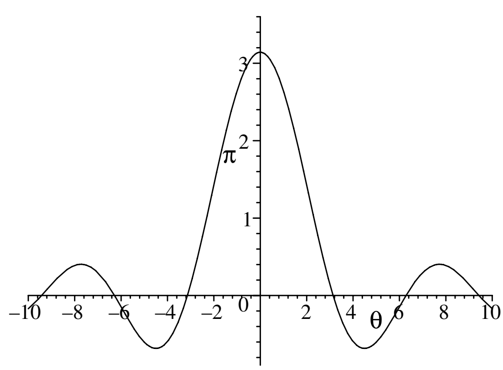

Figure 3 The graph of $\Pi(\theta) = \pi(\sin(\theta))/\theta$

As the terminus of angle θ passes the South Pole we will measure the circumference in the opposite direction, hence yielding the negative values for Π.

Also, we see that $\lim_{\theta\to\pm\infty}\pi(\sin\theta)/\theta = 0$. As θ grows larger the radius is wrapped once around the sphere for every multiple of 2π, but the absolute value of the circumference will never grow larger than the length of the equator.

The graph of our function suggests that a minimum value for Π occurs near ± 4.5. The standard calculus technique of setting the first derivative of $\Pi(\theta) = \pi(\sin\theta)/\theta$ equal to 0 yields the following results:

$$\Pi'(\theta) = \pi\frac{\theta\cos\theta - \sin\theta}{\theta^2} = 0 \Rightarrow \tan\theta = \theta,$$

so that $\theta \approx \pm 4.493409458$. We thus compute the minimum value to be approximately

$$\Pi(4.493409458) = \pi\frac{\sin 4.493409458}{4.493409458} = -.6824595706.$$

Thus we may conclude that for spherical geometry the ratio of the circumference of a circle to its diameter ranges over the values

$$-.6824595706 \leq \Pi < \pi \approx 3.141592654.$$

If we return to the values given to us in First Kings we may compute the size of the sphere on which the measurements of the molten sea are made. Using the equation $C = 2\pi R \sin r/R$, we solve for R when $C = 30$ and $r = 5$:

$$30 = 2\pi R \sin \frac{5}{R} \quad \text{or} \quad \frac{15}{\pi} = R \sin \frac{5}{R}.$$

We have no closed form solution for this equation, but the function

$$F(R) = R \sin\left(\frac{5}{R}\right) - \frac{15}{\pi}$$

is a continuously differentiable function for $R > 0$, so Newton's method would produce an accurate approximation. This takes only a few seconds using computer programs such as *Maple* or *Mathematica*, and we find that $R \approx 9.549296586$ cubits. Thus a circle at latitude 60° on a sphere of radius 9.549296586 cubits will have a circumference of 30 cubits and a diameter of 10 cubits.

The ratio π in hyperbolic geometry Spherical geometry is not the only alternative to the Euclidean plane. Any smooth two-dimensional surfaces in $\mathbb{R}^3$ might do just as well. Any such surface has an intrinsic measurement of curvature, which gives us an idea of how curved the surface is at each point. The curvature, or more precisely the Gaussian curvature, is computed as the product of two other quantities called the *principal curvatures* at a point. These principal curvatures are the maximum and minimum curvatures of the collection of one-dimensional arcs through that point. For a circle the curvature is $1/R$ where $|R|$ is the radius. We comment that R may be positive or negative depending as to whether we make the measurement from a vantage point inside the circle or outside the circle. Since the curvature of every arc on a sphere through any given point is $1/R$ where $|R|$ is the radius of the sphere, we have the curvature of the sphere to be the constant $K = 1/R^2$, which is always positive.

It is possible for surfaces to have negative curvature. The saddle point of a hyperbolic paraboloid is such an example. Since the surface is curving in a concave fashion in one direction and a convex fashion in the other direction, the maximal principal curvature will be positive and the minimal principal curvature will be negative. The Gaussian curvature at the saddle point is thus the product of a negative value and a positive value, which must be negative. O'Neill [**3**] is one standard reference.

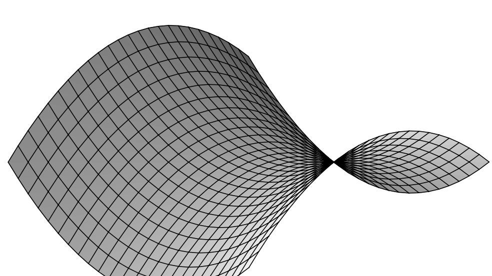

Figure 4 The hyperbolic paraboloid

We may now consider a space where at every point the principal curvatures are $1/R$ and $-1/R$, and hence the Gaussian curvature is the negative value $K = -1/R^2 = 1/(iR)^2$. We may consider this space to be a pseudo-sphere with an imaginary radius. The geometry on this space is known as hyperbolic geometry.

We can replace R with iR in our development of the function $\Pi(\theta)$ to get the corresponding formulas for hyperbolic geometry.

$$C = 2\pi i R \sin \frac{r}{iR} \quad \text{where } \frac{r}{R} = \theta.$$

Thus, we get a formula reminiscent of the spherical ratios,

$$C = 2\pi i r \frac{\sin \frac{\theta}{i}}{\theta} = 2\pi i r \frac{\sin(-i\theta)}{\theta}.$$

Apply the identity $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$, to give us

$$C = 2\pi i r \frac{e^{-i^2\theta} - e^{i^2\theta}}{2i\theta} = 2\pi r \frac{e^{\theta} - e^{-\theta}}{2\theta} \equiv 2\pi r \frac{\sinh\theta}{\theta}.$$

Thus

$$\Pi(\theta) = \frac{C}{2r} = \pi \frac{\sinh\theta}{\theta}.$$

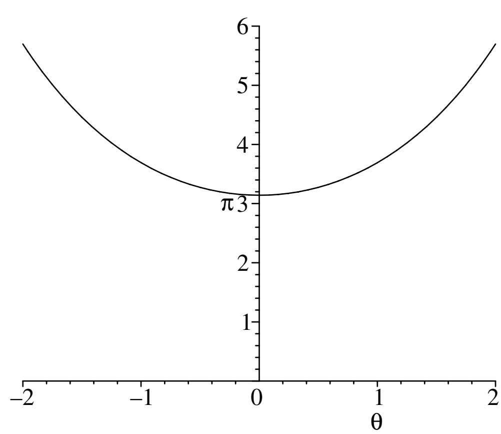

Figure 5 The graph of $\Pi(\theta) = \pi(\sinh\theta)/\theta$

The graph of $\Pi(\theta) = \pi(\sinh\theta)/\theta$ in FIGURE 5 reveals that Π takes on values greater than π in hyperbolic geometry. Easy limit computations produce

$$\lim_{\theta\to 0} \pi \frac{\sinh\theta}{\theta} = \pi \quad \text{and} \quad \lim_{\theta\to\pm\infty} \pi \frac{\sinh\theta}{\theta} = +\infty.$$

So we may conclude that, in hyperbolic geometry, Π takes on all values greater than π.

Conclusion The ancient Hebrews were certainly unaware of alternate geometries and were more concerned with the spiritual aspects of their lives than mathematical precision. Since it is highly unlikely that they would choose a sphere of approximately 9 meters in diameter on which to make their measurements, we can rationally conclude the discrepancies between First Kings and Archimedes is due to a very coarse approximation. But it is entertaining to realize that these measurements can be made exact by using the appropriate geometry.

Archimedes did not know the formal limit concept we use today, but he most surely knew the intuitive concept. Today the exact value of π is known to be the limit of the sequence produced by Archimedes. It is interesting to note that π is the limit of the ratio of circumferences of circles to their diameters in both the spherical and hyperbolic geometries. But this should not be surprising, since the limit is taken as the central angle approaches 0. If we imagine that the diameter of the circle is held constant, then the radius of the sphere or pseudo-sphere must approach infinity and the curvature approaches 0. Thus the Euclidean plane can be thought of as a sphere or pseudo-sphere with curvature 0.

Using our three geometries, π can be assigned any positive real value that you want, and even some negative values. We find it compelling to ponder the possibility of a geometry that would allow all negative values.

REFERENCES

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circle. Thus, no points on the curved parts of the convex hull belong to any of the arcs whose lengths we are summing. Because the curved parts of the convex hull can be translated together to form a unit circle, their total length is 2π. Thus, our bound is improved to $2(n-1)\pi$. Combining this with our previous information, we have

$$\sum_{1\le i<j\le n} \frac{8}{O_i O_j} < 2(n-1)\pi.$$

Dividing by 8 yields the desired inequality.