Math Team Probset 1
October 13, 2005

1) In the figure above, $AB = 20$, $AC = 12$, $AD = BD$, angles ACB and ADE are right angles. Find the area of the quadrilateral $ADEC$.

2) Find all solutions (x, y) (including complex ones) to this system of equations:

$$x^2 + y^2 + x + y = 8$$
$$xy + x + y = 5$$

3) If the pth term of an arithmetic progression is q and the qth term is p, where $p \neq q$, what is the $(p + q)$th term? Note: a sequence $a_0, a_1, a_2, \cdots$ is an arithmetic progression if the difference between consecutive terms is a constant: $a_1 - a_0 = a_2 - a_1 = \cdots$.

4) Given a set of 51 points in a unit square, show that there is always a set of three of those points that lie interior to a circular disk of radius $1/7$.

5) Evaluate the following integral and explain your answer:

$$\int_{-\pi}^{\pi} (2 + x\cos(x^3))\, dx$$

6) Let f be a differentiable function on $[0, 1]$, with $f(0) = f(1) = 0$. Prove that there is a point u in the open interval $(0, 1)$ with

$$|f'(u)| = 4\int_0^1 |f(x)|\, dx$$

7) The Prof Haller Problem:
Consider a game with the 5 boxes drawn below. The box labelled S is the start box; if you land in the box labelled W you win, if you land in the box labelled L you lose. You move by rolling a fair six-sided die: if you roll a 4 or 6 you move one box to the right, if you roll 1, 2, 3, or 5 you move one box to the left. Once you win or lose, the game is over.

a) You roll the die four times (if necessary). What's the probability that you win? What's the probability that you lose? What's the probability that you neither won nor lost? If you neither won nor lost, in which square do you end up after four rolls of the die?
b) You roll the die six times (if necessary). What's the probability that you win? What's the probability that you lose? What's the probability that you neither won nor lost? If you neither won nor lost, in which square do you end up after six rolls of the die?
c) If the game is allowed to go on until you either win or lose, what's the probability that you win? that you lose? that the game never ends? Are the two statements below contradictory? Explain.

1. The probability of the game never ending is zero.

2. It is possible that the game never ends.