Problem Solving Set 1, Spring 2005
Calculus

1) Prove that the function below is constant. What constant?

$$f(x) = \arctan\left(\frac{x+1}{x-1}\right) + \arctan(x)$$

2) Suppose $y = f(x)$ is a curve that always lies above the x axis and never has a horizontal tangent, and f is differentiable at all real numbers. For what value of y is the rate of change of y^5 with respect to x eighty times the rate of change of y with respect to x?

3) The figure shows a region consisting of all points inside a square that are closer to the center than to the sides of the square. Find the area of the region.

4) Let $f(x) = \int_0^x e^{-t^2} dt$. Given that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, what is $\int_0^{\infty} e^{-x^2+f(x)}\ dx$?

5) In the figure there are infinitely many circles approaching the vertices of an equilateral triangle, each circle touching other circles and sides of the triangle. If the triangle has sides of length 1, find the total area occupied by the circles.

6) Suppose f and g are differentiable functions so that $f(g(x)) = x$ and $f'(x) = 1 + [f(x)]^2$. Find $g'(x)$.

7) Find the following limit:

$$\lim_{x\to 0} \frac{\sin(\tan(x)) - \tan(\sin(x))}{\arcsin(\arctan(x)) - \arctan(\arcsin(x))}$$

8) **Putnam, 1980**
For which real numbers c does the following hold, for all x?

$$\frac{e^x + e^{-x}}{2} \leq e^{cx^2}$$

9) **Putnam, 2000**
Show that the following integral exists:

$$\int_0^{\infty} \sin(x)\sin(x^2)\ dx$$

10) **MAA Problem 11,111, from vol 111, no 9; November 2004**
Let f and g be nonconstant, continuous, periodic functions mapping $\mathbb{R}$ into $\mathbb{R}$. Is it possible that the function h on $\mathbb{R}$ by $h(x) = f(xg(x))$ is periodic?