Problem Solving Set 2, Spring 2005
Geometry

1) A *Gaussian integer* is a complex number $a + ib$ where a, b are integers. A *Pythagorean triple* is three positive integers a, b, c with $a^2 + b^2 = c^2$. If we take the Gaussian integer $2 + 3i$ and square it, we get $(2 + 3i)^2 = -5 + 12i$, and that 5 and 12 are the first two terms of the Pythagorean triple $5, 12, 13$. Is that always the case? Prove or find a counter-example to the following assertion: For every Gaussian integer $a + ib$ with $|a| \neq |b|$ and $ab \neq 0$, the real and imaginary parts of $(a + ib)^2$ are, in absolute value, the first two terms of a Pythagorean triple.

2) Let $p(x)$ be a polynomial of odd degree. Does every point in the plane lies on a tangent line to $p(x)$? Can you show this without calculus?

3) Let A, B, C, D be four distinct points on a line, in that order. The circles with diameter AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersectst the circle with diameter BD at B and N. Prove that the lines AM, DN, and XY all meet at one point.

4) On a small square billiard table with sides of length 2 feet, a ball is played from the center and, after rebounding off the sides several times, goes into a cup at one of the corners. Prove that the total distance travelled by the ball is *not* an integer number of feet.

5) An isosceles triangle with an inscribed circle is labeles as shown in the circle. Find an expression, in terms of the angle α and the length a, for the area of the curvilinear triangle bounded by sides AB, AC, and the arc BC.

6) From a point D in the hypotenuse BC of a right triangle ABC, perpendiculars DE and DF are drawn to AC and AB, respectively. Determine the position of D for which EF has minimum length.

7) Show that the sum of the angles in a spherical triangle is greater than 180 degrees.

8) **Putnam, 1985**

Let T be an acute triangle. Inscribe a rectangle R in T, with one side along a side of T. Then inscribe a rectangle S in the triangle formed by the side of R opposite the side on the boundary of T, and the other two sides of T, with one side along the side of R. Let $A(R)$ be the area of R; let $A(S)$ be the area of S; let $A(T)$ be the area of T. Find the maximum value, or show that no maximum exists, of:

$$\frac{A(R) + A(S)}{A(T)}$$

where T ranges over all triangles, and R, S over all rectangles as above.

9) **Putnam, 2004**

Let T_1 be a triangle with side lengths a_1, b_1, c_1 and area A_1. Likewise, let T_2 be a triangle with side lengths a_2, b_2, c_2 and area A_2. Suppose the following: $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$, and T_2 is an acute triangle. Does it follow that $A_1 \leq A_2$?

10) **MAA Problem 11,122, from vol 111, no 10; December 2004**

A positive integer is *perfect* if it is the sum of its proper divisors. A *Pythagorean triangle* is a right triangle with integer sides. Prove that the legs of a Pythagorean triangle cannot both be perfect numbers. Is it possible that one of the legs and the hypoteneuse of a Pythagorean triangle are both perfect numbers?