Problem Solving Set 3, Spring 2005
Pigeon Hole Principle

1) A 9×9 chessboard has two squares from opposite corners and its central square removed (so that 3 squares on the same diagonal are removed, leaving 78 squares). Is it possible to cover the remaining squares with dominoes, where each domino covers two adjacent squares? Justify your answer.

2) Five points are situated inside an equilateral triangle whose side has length one unit. Show that two of them may be chosen which are less than 1/2 unit apart. What if the equilateral triangle is replaced by a square whose sides are 1 unit?

3) Does the set $\{1, 2, ..., 3000\}$ contain a subset A of 2000 numbers with the following property: if $x \in A$, then $2x \notin A$?

4) S is the set $\{1, 2, 3, ..., 1,000,000\}$. Show that for any subset A of S with 101 elements we can find 100 distinct elements x_i of S, such that the sets $x_i + A$ are all pairwise disjoint. (Note that $x_i + A$ is the set $\{a + x_i | a \in A\}$).

5) Integers are placed in each of the 441 cells of a 21×21 size array. Each row and each column has at most 6 different integers in it. Show that some integer is in at least 3 rows and 3 columns.

6) In any partition of the first $2N$ natural numbers into increasing and decreasing sequences of N numbers each, the sum of the absolute values of the differences of the corresponding members of the two sequences is always N^2.

7) Let n and k be non-negative integers, with $n \geq k$. The binomial coefficient $\binom{n}{k}$ is defined by $\binom{n}{k} = n!/k!(n-k)!$. (Recall that $0! = 1$. If $n < k$, then it is convenient to define $\binom{n}{k} = 0$.).

(a) For what values of n and k is $\binom{n}{k}$ odd? Find as simple and elegant a criterion as possible.

(b) More generally, given a prime p, find a simple and elegant description of the largest power of p dividing $\binom{n}{k}$.

8) **Putnam, 1978**

Let A be a set of any twenty integers chosen from the arithmetic progression $1, 4, 7, \cdots, 97, 100$. Prove that there are at least two distinct integers in A whose sum is 104.

9) **Putnam, 1995**

For a partition π of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that, for any two partitions π and π', there are two distinct numbers x and y in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ so that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. Note: a *partition* of a set S is a collection of non-empty disjoint subsets (parts) of S whose union is S; these subsets are also called *equivalence classes*.

10) **MAA Problem 11,134, from vol 112, no 2; February 2005**

Fix primes p and q. Show that there are at most 6 integers x so that the area of the triangle with side lengths p, q, x is an integer.