Hints and Comments on Problem Set 3
The Pigeonhole Principle

Comments. The Pigeonhole Principle is pretty straightforward: if I've got more pigeons than holes, I have to cram at least two pigeons into at least one hole. You do have to be a little careful though; if I have 12 pigeons and 7 holes, the Principle allows me to conclude that I have at least 2 pigeons in at least one hole, but doesn't allow me to conclude any more than that. I could have 1 pigeon in each of 2 holes, and 2 pigeons in each of 5 holes, or I could've crammed all 12 pigeons into 1 hole. Mathematically, the Pigeonhole Principle is:

If N and M are finite sets with $|N| > |M|$, then there is no one to one function from N to M.

In Pigeonhole Principle problems, the trick is to figure out what are the holes and what are the pigeons, and show you've got more pigeons than holes. Or, more mathematically, figure out what M and what N are, and then show $|N| > |M|$.

The Pigeonhole Principle can be generalized in a number of ways.

1. We can ask if we have to have at least 2 (or 3, or 4, or more) pigeons in at least one pigeonhole. For example, if I have 23 pigeons and 7 holes, then I can put 3 pigeons in each hole, but I'd have 2 left over. And so at least one hole will have to have at least 4 pigeons.

2. We might know how many pigeonholes we have, and then have to count the pigeons. If I have 7 pigeonholes, and my pigeons are even numbers from 1 to 10, do I have enough pigeons to fill all my pigeonholes?

3. We might know how many pigeons we have, and we might know how many pigeonholes we have; the problem might be that each pigeon might consume more than one pigeonhole (they might be very fat pigeons). Maybe we'll run out of pigeonholes before we run out of pigeons? Problem 1 can be thought of in this sense–39 pigeons (the dominoes), but each domino that is laid down eliminates more than 1 possible pair of adjacent squares (the pigeonholes); will I run out of adjacent squares before I run out of dominoes?

WARNING! In the following, I include my best guess about what the pigeons and pigeonholes might be. In all fairness, I must tell you that I am terrible at these sorts of problems; I don't really have any intuition for them. So these best guesses could very well be terribly wrong. Use them at your own risk.

Anyway, on to the hints.

Hints on Problems

3. To restate this question: can you pick 2000 numbers, out of the first 3000, so that none of the numbers you pick is twice another number you've picked? One way to attempt this is the following.

1. The set $\{1501, \cdots 3000\}$ is a set of 1500 numbers, none of which is twice of any other. So I can start with $A = \{1501, \cdots 3000\}$

2. All the numbers in the set $\{751, 1500\}$ have doubles in the set $\{1501, \cdots 3000\}$, so I can't include them in A.

3. All the numbers in the set $\{376, 750\}$ have doubles in the set $\{751, 1500\}$, which are not in A, so I can throw this set into A, and now $A = \{376, 750\} \cup \{1501, \cdots 3000\}$. And this set

satisfies the condition; if $x \in A$, then $2x \notin A$. But A has only $375 + 1500 = 1875$ numbers in it.

4. All the numbers in the set $\{188, 375\}$ have doubles in the set $\{376, 750\}$, which are in A, and so I can't include any of these numbers in A.

5. All the numbers in the set $\{94, 187\}$ have doubles in the set $\{188, 375\}$, which are not in A, so I can throw this set into A. Now $A = \{94, 187\} \cup \{376, 750\} \cup \{1501, \cdots 3000\}$, which is a set that satisfies the condition. But this set has only $94 + 375 + 1500 = 1969$ numbers in it.

6. All the numbers in the set $\{47, 93\}$ have doubles in the set $\{94, 187\}$, so I can't include any of these numbers in A.

7. All the numbers in the set $\{24, 46\}$ have doubles in the set $\{47, 93\}$, which are not in A, so I can throw this set into A. Now, $A = \{24, 46\} \cup \{94, 187\} \cup \{376, 750\} \cup \{1501, \cdots 3000\}$, which is a set that satisfies the condition. But this set has only $23 + 94 + 375 + 1500 = 1992$ numbers in it.

8. All the numbers in the set $\{12, 23\}$ have doubles in the set $\{24, 46\}$, and so I can't include any of these numbers in A.

9. All the numbers in the set $\{6, 11\}$ have doubles in the set $\{12, 23\}$, which are not in A, so I can throw this set into A. This makes $A = \{6, 11\} \cup \{24, 46\} \cup \{94, 187\} \cup \{376, 750\} \cup \{1501, \cdots 3000\}$, which is a set that satisfies the condition. But this set has only $6 + 23 + 94 + 375 + 1500 = 1998$ elements in it.

10. Of the numbers $1, 2, 3, 4, 5$, I can't include $3, 4, 5$ in A, since their doubles are already in A. I can include only one of $1, 2$; I'll pick 2. So, at the end of the process, I have $A = \{2\} \cup \{6, 11\} \cup \{24, 46\} \cup \{94, 187\} \cup \{376, 750\} \cup \{1501, \cdots 3000\}$, a set with $1 + 6 + 23 + 94 + 375 + 1500 = 1999$ elements in it.

So this process got us very very close to an answer, but not quite. You might want to ask Dr Walk about it; I gave him this problem while we were sitting through the Faculty Senate meeting Tuesday afternoon, and he was working on it for awhile.

Hard to say what the pigeons and pigeonholes are here; I was thinking that the pigeons are pairs of numbers $(x, 2x)$ and the pigeons are individual numbers, but I'm not really sure.

4. Here's an example. Set $A = \{1, 2, \cdots, 100, 101\}$. Pick $x_1 = 1000$. Then the notation $x_1 + A$ refers to the set which is obtained by adding $x_1 = 1000$ to each element of A, and so:

$$\begin{aligned} x_1 + A &= \{1000 + 1, 1000 + 2, \cdots, 1000 + 100, 1000 + 101\} \\ &= \{1001, 1002, \cdots, 1100, 1101\} \end{aligned}$$

If we pick $x_2 = 900$, then:

$$\begin{aligned} x_2 + A &= \{900 + 1, 900 + 2, \cdots, 900 + 100, 900 + 101\} \\ &= \{901, 902, \cdots, 1000, 1001\} \end{aligned}$$

This is a bad choice of x_1 and x_2, since the sets $x_1 + A$ and $x_2 + A$ overlap. But! there are 100 good choices for $x_1, \cdots, x_{100}$ so that none of the sets $x_1 + A, \cdots, x_{100} + A$ overlap.

Hard to tell what the pigeons and pigeonholes would be here; my best guess would be that the pigeons are the sets $x_i + A$ and the pigeonholes are the possible 101 element subsets of S.

5 There is exactly one integer in each cell. Each column thus must have at least one integer repeated four times in the cells of that column.

My best guess about is that the cells are the pigeonholes and the numbers are the pigeons; there's a lot of identical pigeons though (the number 7, for example, might show up in lots of different cells).

6. As an example, let $N = 5$, and $2N = 10$. Then the first $2N = 10$ natural numbers are: $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$. Pick any $N = 5$ of them: say $1, 3, 6, 8, 9$, and arrange them from smallest to biggest. Write the other $N = 5$ numbers from biggest to smallest: $10, 7, 5, 4, 2$. So we can subtract the lists:

$$|1 - 10| = 9$$
$$|3 - 7| = 4$$
$$|6 - 5| = 1$$
$$|8 - 4| = 4$$
$$|9 - 2| = 7$$

And when I add all these differences, I will get $N^2 = 25$:

$$9 + 4 + 1 + 4 + 7 = 25$$

You are asked to show that this sort of thing happens no matter what N I choose, and no matter what sequence of numbers I choose from the first $2N$ natural numbers.

My best guess is that the pigeonholes are the sizes of the differences, and, by the way we construct the sequences, there is some relationship between them (N^2 - the sum of the first $(N - 1)$ pigeonholes = the last pigeonhole). I guess that makes the pigeons the pairs of numbers whose differences we take.

7 It's not clear exactly what's meant by 'elegant description.' It should be some fairly straightforward statement, either an equation, or a brief algorithm, or a brief statement. Something with lots of sums and 'if...then...' statements is probably not elegant. This is a fairly hard problem; there are lots of results (including both of these) at MathWorld. Remember that the binomial coefficients are the numbers in Pascal's triangle; you might want to try writing down the first 20 or 25 rows of Pascal's triangle, using blue for the odd numbers and red for the even numbers, and see what patterns show up.

This is not really a pigeonhole principle problem; I just like Pascal's triangle. And if you go to Pascal's Triangle at MathWorld, they'll show you how to get the Fibonacci numbers from the triangle! Neat!

9 A partition of a set is just chopping it up into disjoint subsets. If we let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be our set, then a partition of A would be:

$$\pi = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8, 9\}\}$$

So this partition π is a list of three subsets of A, and each element of A is in exactly one of the three subsets. The notation $\pi(x)$ means the size of the subset that the element x is in, and so:

$$\begin{array}{lll} \pi(1) = 3 & \pi(2) = 3 & \pi(3) = 3 \\ \pi(4) = 2 & \pi(5) = 2 & \pi(6) = 4 \\ \pi(7) = 4 & \pi(8) = 4 & \pi(9) = 4 \end{array}$$

Another partition of A is:

$$\pi' = \{\{1\}, \{2, 3, 4, 5, 6\}, \{7\}, \{8\}, \{9\}\}$$

This is a list of 5 subsets of A, and each element of A is in exactly one of the five subsets. Here, $\pi'(x)$ means the size of the subset in this partition that the element x is in, and so:

$$\begin{array}{lll} \pi'(1) = 1 & \pi'(2) = 5 & \pi'(3) = 5 \\ \pi'(4) = 5 & \pi'(5) = 5 & \pi'(6) = 5 \\ \pi'(7) = 1 & \pi'(8) = 1 & \pi'(9) = 1 \end{array}$$

Note an interesting thing that happened here:

$$\begin{array}{ll} \pi(2) = 3 & \pi(3) = 3 \\ \pi'(2) = 5 & \pi'(3) = 5 \end{array}$$

I can find two numbers in A that end up in the same size subset in both partitions. Note also:

$$\begin{array}{ll} \pi(8) = 4 & \pi(9) = 4 \\ \pi'(8) = 1 & \pi'(9) = 1 \end{array}$$

Both 8 and 9 ended up in the same size subsets in both partitions.

This always happens. No matter what two partitions of this set you choose, there will be at least one pair of numbers that end up in the same size subsets in both partitions. That is what you are asked to show.

My best guess is that the possible sizes of the subsets are the pigeonholes, and the numbers are pigeons.

10 There's a very nice formula called (I think) Herron's formula for the area of a triangle in terms of its semi-perimeter. So: if P is the perimeter of a triangle, then $S = (1/2)P$ is its semi-perimeter. If a, b, c are the three sides of the triangle, then the area A is:

$$A = \sqrt{S(S-a)(S-b)(S-c)}$$

So imagine then that we know two sides of our triangle, and that they are both prime: $a = p$ and $b = q$, where p, q are prime numbers. Can we choose integers x to be the third side, so that the area A is also an integer? How many possible integers are there? I tried this with $p = 2$ and $q = 3$; there were no solutions (not entirely surprising). There might be other ways to attack this, other than Herron's formula, but, on first impression, this seemed most promising. This is not a Pigeonhole Principle Problem, by the way.