Problem Solving Set 4, Spring 2005
Induction

1) Let M be the matrix below. Find a formula for M^n, and prove that it is true for all natural numbers n. Put $n = -1/2$ into your formula, and interpret the resulting matrix. Is it in some sense $M^{-1/2}$?

$$M = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

2) Chocolate bars come impressed with little lines on them, so that they're easy to break into bite size pieces. Show that if you break a bar down into the bite size pieces by only breaking along the lines, then, no matter how you proceed, it always takes the same number of breaks, and that this number of breaks is equal to the number of bite size pieces minus one.

3) Let A, B be square matrices of the same size. Prove that $\det(A)\det(B) = \det(AB)$.

4) Prove the following equation, for all n. (Assume the denominators are all nonzero).

$$\frac{1}{\sin(2x)} + \frac{1}{\sin(4x)} + \cdots + \frac{1}{\sin(2^n x)} = \cot(x) - \cot(2^n x)$$

5) You have an unlimited supply of 21 cent stamps and 64 cent stamps. Show that you cannot get exactly 12.59 dollars worth of postage using these stamps, but, for any dollar value greater than 12.59, you can.

6) Let n straight lines lie in the plane, so that no two are parallel and no three lines have a common point. Into how many regions do these lines divide the plane? Show that you can color these regions black and white so that no two regions that share a border are the same color.

7) Prove, for all n:

$$\frac{1}{n^2} + \left(\frac{1}{n} + \frac{1}{n-1}\right)^2 + \left(\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2}\right)^2 + \cdots +$$
$$\left(\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} \cdots + \frac{1}{n-(n-2)} + 1\right)^2 = 2n - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$

8) **Putnam, 2002**
Let k be a fixed positive integer. The nth derivative of $1/(x^k - 1)$ has the form $P_n(x)/(x^k - 1)^{n+1}$, where $P_n(x)$ is a polynomial. Find $P_n(1)$.

9) **Putnam, 2003**
Show for each positive integer n:

$$n! = \prod_{i=1}^{n} \operatorname{lcm}(1, 2, 3, \cdots, \lfloor n/i \rfloor)$$

where $\lfloor x \rfloor$ is the greatest integer less than x.

10) **MAA Problem 11,103, from vol 111, no 8; October 2004**

Prove the following equation for every positive integer n:

$$\sum_{k=1}^{n} \frac{1}{k\binom{n}{k}} = \frac{1}{2^{n-1}} \sum_{k=1, k \text{odd}}^{n} \frac{\binom{n}{k}}{k}$$