Problem Solving Set 5, Spring 2005
The Floor Function

1) Let n be a positive integer. Find a formula for

$$\int_1^n \frac{1}{\lfloor x^2 \rfloor}\, dx$$

Your formula may include a summation, but not an integration.

Does the following integral converge or diverge?

$$\int_1^\infty \frac{1}{\lfloor x^2 \rfloor}\, dx$$

2) Find a closed form expression for

$$f(n) = \sum_{k=1}^{n^2} \frac{n - \lfloor \sqrt{k-1} \rfloor}{\sqrt{k} + \sqrt{k-1}}$$

3) For r a positive number, let r be the fractional part of r; thus $r = \lfloor r \rfloor + F(r)$. Find a positive number r so that:

$$F(r) + F\left(\frac{1}{r}\right) = 1$$

4) Find all integers n so that the following is true:

$$\lfloor \sqrt[4]{1} \rfloor + \lfloor \sqrt[4]{2} \rfloor + \lfloor \sqrt[4]{3} \rfloor + \cdots + \lfloor \sqrt[4]{n} \rfloor = 2n$$

5) As in problem 3, $F(x)$ is the fractional part of the number x. Let x be a real number, and $n \geq 3$ an integer, and suppose $F(x) = F(x^2) = F(x^n)$. Show that x is an integer.

6) Find the area between the graphs of the following two functions, for $0 \leq x \leq 24036$.

$$f(x) = x^2 - \frac{1}{2}x$$
$$g(x) = x\lfloor x \rfloor$$

7) If n is a positive integer, show:

$$\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+2} \rfloor$$

8) **Putnam, 1986**

Find the units digit of

$$\left\lfloor \frac{10^{20000}}{10^{100} + 3} \right\rfloor$$

9) **Putnam, 1998**

Find necessary and sufficient conditions on positive integers n and m so that

$$\sum_{i=1}^{mn-1} (-1)^{\lfloor m/i \rfloor + \lfloor n/i \rfloor} = 0$$

10) **Not an MAA problem, but maybe should be**
Does the following series converge? If so, to what?

$$\sum_{n=1}^{\infty} \frac{2^{\lfloor \sqrt{n} \rfloor} + 2^{-\lfloor \sqrt{n} \rfloor}}{2^n}$$

A similar problem, from the 2001 Putnam:
If $< n >$ is the integer closest to $\sqrt{n}$, show that

$$3 = \sum_{n=1}^{\infty} \frac{2^{<n>} + 2^{-<n>}}{2^n}$$