Problem Solving Set 7, Spring 2005
Algebra

1) Let S be a set closed under the binary operation $*$, with the following properties:

1. there is $e \in S$ so that $a * e = e * a = a$ for all $a \in S$
2. $(a * b) * (c * d) = (a * c) * (b * d)$ for all $a, b, c, d \in S$

Prove or disprove:

1. $*$ is associative
2. $*$ is commutative

2) Let S be a set closed under the binary operation $*$ with the following property:

$$(w * x) * (y * z) = w * z$$

for all $w, x, y, z \in S$.

Show:

1. if $a * b = c$, then $c * c = c$
2. if $a * b = c$, then $a * x = c * x$, for all $x \in S$

3) Let A, B be $n \times n$ matrices. If $ABAB = 0$, does $BABA = 0$?

4) Let R be a finite ring, not necessarily commutative, containing an element r which is not a divisor of zero (ie there is no $s \in R$, $s \neq 0$, with $rs = 0$ or $sr = 0$). Prove R has a multiplicative identity.

5) Show that $\mathbb{Z}_p$ is a field, but that $\mathbb{Z}_n$ is not, if n is not prime.

6) Let $R = \mathbb{Z}_6$. In the polynomial ring $R[x]$, is factorization $x^2 + 3x + 2 = (x+1)(x+2)$ unique?

7) Over the field $\mathbb{Z}_p$, show that

$$x^p - x = x(x-1)(x-2)\cdots(x-(p-1))$$

Can you factor $x^{p^n} - x$ over $\mathbb{Z}_p$?

8) **Putnam, 1991**
Let p be an odd prime; let $\mathbb{Z}_p$ be the field of integers mod p. How many elements are in the set:

$$\{x^2 \mid x \in \mathbb{Z}_p\} \cap \{y^2 + 1 \mid y \in \mathbb{Z}_p\}$$

9) **Putnam, 2002**

Let p be a prime number. Prove that the determinant below:

$$\begin{vmatrix} x & y & z \\ x^p & y^p & z^p \\ z^{p^2} & y^{p^2} & z^{p^2} \end{vmatrix}$$

is congruent modulo p to a product of polynomials of the form $ax+by+cz$, where a, b, c are integers.

10) **Kumiko's Origami Problem**

What's the group of symmetries of one of Kumiko's origami things?
And a grammar question: are the following equivalent?

1. Kumiko's Origami Problem
2. The Problem of Kumiko's Origami
3. The Origami Problem of Kumiko
4. The Problem of the Origami of Kumiko