Problem Solving Set 8, Spring 2005
Matrices

1) Evaluate this determinant:

$$\begin{vmatrix} a_1^2+x & a_1a_2 & a_1a_3 & \cdots & a_1a_n \\ a_2a_1 & a_2^2+x & a_2a_3 & \cdots & a_2a_n \\ a_3a_1 & a_3a_2 & a_3^2+x & \cdots & a_3a_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_na_1 & a_na_2 & a_na_3 & \cdots & a_n^2+x \end{vmatrix}$$

2) A square matrix is **nilpotent** if $A^k = 0$ for some positive integer k. Prove that if A and B are nilpotent matrices, and $AB = BA$, then $A + B$ is a nilpotent matrix.

3) Let A be a non-zero square matrix with the property that $A^3 = 0$, where 0 is the zero matrix, but with A otherwise being arbitrary. Express $(\mathrm{Id}_n - A)^{-1}$ as a polynomial in A, where Id_n is the identity matrix.

4) Let $\vec{a}, \vec{b}, \vec{c}$ be linearly dependent vectors. Show:

$$\begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{a}\cdot\vec{c} \end{vmatrix} = 0$$

5) Find real numbers c_1 and c_2 so that

$$\mathrm{Id}_2 + c_1M + c_2M^2 = 0$$

where Id_2 is the 2×2 identity matrix, and 0 is the 2×2 zero matrix, and

$$M = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$$

6) Let A be a 3×3 matrix in which each element is 0 or 1. Prove that $|A|$ cannot be 3 or -3. Find all possible values of $|A|$.

7) Let Id_2 be the 2×2 identity matrix. Let A, B, C be 2×2 matrices. Let M, N be 4×4 matrices as follows:

$$M = \begin{bmatrix} \mathrm{Id}_2 & A \\ B & C \end{bmatrix} \qquad N = \begin{bmatrix} \mathrm{Id}_2 & B \\ A & C \end{bmatrix}$$

Prove or disprove: if M is invertible, then N is invertible.

8) **Putnam, 1991**
Let A and B be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, is it possible that $A^2 + B^2$ is invertible?

9) **Putnam, 1986**
Suppose A, B, C, D are $n \times n$ matrices with real entries and the following properties:

1. AB^t and CD^t are symmetric
2. $AD^t - BC^t = \mathrm{Id}_n$, where Id_n is the $n \times n$ identity matrix.

Prove that $A^tD + C^tB = \mathrm{Id}_n$.

10) **Linear Differential Equations**
a) Define M as follows:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

Let t be a real variable and define:

$$e^{Mt} = \mathrm{Id}_2 + Mt + \frac{1}{2!}(Mt)^2 + \frac{1}{3!}(Mt)^3 + \cdots + \frac{1}{n!}(Mt)^n + \cdots$$

Show:

$$e^{Mt} = \begin{bmatrix} e^t & 0 \\ 0 & e^{4t} \end{bmatrix}$$

b) Define N as follows:

$$N = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

What are the eigenvalues of N? What are the corresponding eigenvectors? Let C be matrix whose columns are eigenvectors of N. What is $C^{-1}NC$? Show that

$$e^{Nt} = C^{-1}e^{Mt}C$$

c) Let $\vec{x} = (x_0, y_0)$. In what sense is $e^{Mt}\vec{x}$ a solution to the differential equation

$$\frac{dx}{dt} = x$$
$$\frac{dy}{dt} = 4y$$

with initial condition $(x(0), y(0)) = (x_0, y_0)$?
d) Let $\vec{x} = (x_0, y_0)$. In what sense is $e^{Nt}\vec{x}$ a solution to the differential equation

$$\frac{dx}{dt} = 2x + 2y$$
$$\frac{dy}{dt} = x + 3y$$

with initial condition $(x(0), y(0)) = (x_0, y_0)$?