Math Team First Problem Set, Fall 2006
Trigonometry and Combinatorics

(1) Prove the following[1]:

$$\sec\left(\frac{\pi}{7}\right)\sec\left(\frac{2\pi}{7}\right)\sec\left(\frac{3\pi}{7}\right) = 8$$

Find a suitable generalization, along the lines of

$$\sec\left(\frac{\pi}{2n+1}\right)\sec\left(\frac{2\pi}{2n+1}\right)\cdots = ?$$

(2) What's the coefficient of x^9 in the expansion of $(2x^2 - x^{-3})^12$?

(3) A card shuffling machine always rearranges cards in the same way relative to the order in which they are given to it. The thirteen spades arranged in the order

$$A,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ J,\ Q,\ K$$

are put into the machine, shuffled, and then the shuffled cards are put into the machine and shuffled again. If at this point the order of the cards is

$$3,\ K,\ 10,\ 2,\ Q,\ 9,\ 4,\ J,\ 8,\ 6,\ 7,\ A,\ 5$$

what was the order of the cards after the first shuffle?

(4) The sequence of functions $\{un(x)\}$ is defined for real x by $u_1(x) = \cos(x/2)$ and for $n > 1$, $u_n(x) = u_{n-1}(x)\cos(x/2n)$. Thus

$$u_n(x) = \cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{2^2}\right)\cdots\cos\left(\frac{x}{2^n}\right)$$

If $x = 0$, it is clear that $u_n(x) = 1$ for every n. Find[2] $\lim_{n\to\infty} u_n(x)$ as a function of x for $x \neq 0$.

(5) Let n be a positive integer. Let P_n be the probability that in $2n$ tosses of a fair coin exactly n heads occur, and P_{2n} the probability that in $4n$ tosses of a fair coin exactly $2n$ heads occur. Which is larger, P_n or P_{2n}? Does the answer depend on n, and if so, how? Defend your answer.[3]

(6) Let a and b be real numbers, with $-1 < a < b < 1$, and let $f(x)$ be a function in the class C^1 (i.e., the derivative $f'(x)$ exists[4] and is continuous) on the interval $[a, b]$. Show that

$$\lim n \to \infty \int_a^b f(x)\sin(nx)dx = 0$$

(7) Adolph shuffles a standard 52-card deck and Bertha will draw 3 cards at random from it (without replacement). Adolph offers to bet even money that at least one of the cards drawn is a face card. Should Bertha take the bet? Explain. (There are 12 face cards in the deck.)

(8) Prove that[5], for all $x, y \in \mathbb{R}$:

$$\cos(x) + \cos(y) + \sin(x)\sin(y) \leq 2$$

(9) (1947 Putnam, Morning Session Problem 4) A coast artillery gun can fire at any angle of elevation between 0 and $\pi/2$ in a fixed vertical plane. If air resistance is neglected and the muzzle velocity v_0 is constant, determine the set H of points in the plane and above the horizontal that can be hit.

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[1] Hint: Recall that $2\sin(x)\cos(x) = \sin(2x)$ and $\cos(-x) = \cos(x)$
[2] Hint: Find $u_n(x)\sin(x/2^n)$, and see if some sort of induction is useful
[3] Hint: Which is bigger, P_n or P_{n+1}
[4] Hint: if all you know about a function is that it has a derivative, then the only integration technique that might be useful is parts
[5] Hint: consider the vectors $(\cos(x), 1, \sin(x))$ and $(1, \cos(y), \sin(y))$ and their dot product

(10) (1952 Putnam, Afternoon Session Problem 1) A mathematical moron is given two sides and the included angle of a triangle and attempts to use the Law of Cosines: $a^2 = b^2 + c^2 - 2bc\cos(A)$, to find the third side a. He uses logarithms as follows. He finds $\ln(b)$ and doubles it; adds to the the double of $\ln(c)$; subtracts the sum of the logarithms of $2, b, c$, and $\cos(A)$; divides the result by 2; and takes the anti-logarithm. Although his method may be open to suspicion, his computation is accurate. What are the necessary and sufficient conditions on the triangle that this method should yield the correct result?