Math Team Second Problem Set, Fall 2006
Geometry and Numbers

(1) In the given figure, $AB = 20$, $AC = 12$, $AD = DB$, angles ACB and ADE are right angles. Find the area of the quadrilateral $ADEC$.

(2) In "La Geometrie," Descartes gives the following geometric construction of a square root.

> If the square root of GH is desired, I add, along the same straight line, FG equal to unity; then bisecting FH at K, I describe the circle FIH about K as center, and draw from G a perpendicular and extend it to I, and GI is the required root.

The given figure accompanies Descartes' prescription. Assuming, as the figure does, that $GH > 1$, prove that the length GI is the required root.[1]

(3) A rectangle with sides a and b is circumscribed by another rectangle of area m^2. Determine all possible values of m in terms of a and b.[2]

(4) Find all right triangles with perimeter 6 units and with integral area (or prove that none exist).[3]

(5) If the number 7^{7^7} is written out in decimal form, what is the last (rightmost) digit? Defend your answer.[4]

(6) Do there exist integers m and n satisfying $130m + 559n = 52$? If so, find such a pair (m, n); if not, explain.[5]

(7) Find all pairs (x, y) of integers such that $1 + 2002x + 2004y = xy$. Note that 2003 is a prime number.[6]

(8) Suppose $a^4 + b^4 + c^4 + d^4 \leq 1$. Show that[7]

$$\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4} + \frac{1}{d^4} \geq 16$$

[1] Remember the equation for a circle (Descartes did invent analytic geometry, after all).

[2] If θ is an angle in a triangle, what's the area of the triangle in terms of θ?

[3] If a, b are the legs and c the hypotenuse, try to find an equation involving $3(a + b)$

[4] In general, what is the rightmost digit of 7^{10k-1}, regardless of what k is?

[5] When does the Euclidian algorithm guarantee such a solution?

[6] So what happens if you take that equation mod 2003?

[7] Recall that $u^2 + v^2 \geq 2uv$, for all real numbers u, v. This is the Cauchy-Schwarz inequality from Math 312, applied to vectors of the one-dimensional vector space $\mathbb{R}^1$. Or, simply note that $(u-v)^2 \geq 0$, and so $u^2+v^2-2uv \geq 0$.

(9) (1965 Putnam, Morning Session Problem 6) If A, B, C, D are four distinct points such that every circle through A and B intersects (or coincides with) every circle through C and D, prove that the four points are either collinear (all on the same line) or co-cyclic (all on the same circle).

(10) (1966 Putnam, Morning Session Problem 4) Prove that after deleting the perfect squares form the list of positive integers the number we find in the nth position is equal to $n + \{\sqrt{n}\}$, where $\{\sqrt{n}\}$ denotes the integer closest to $\sqrt{n}$.